On the Relative Investment Attractiveness of Equities and Bonds

Paul Kotschy

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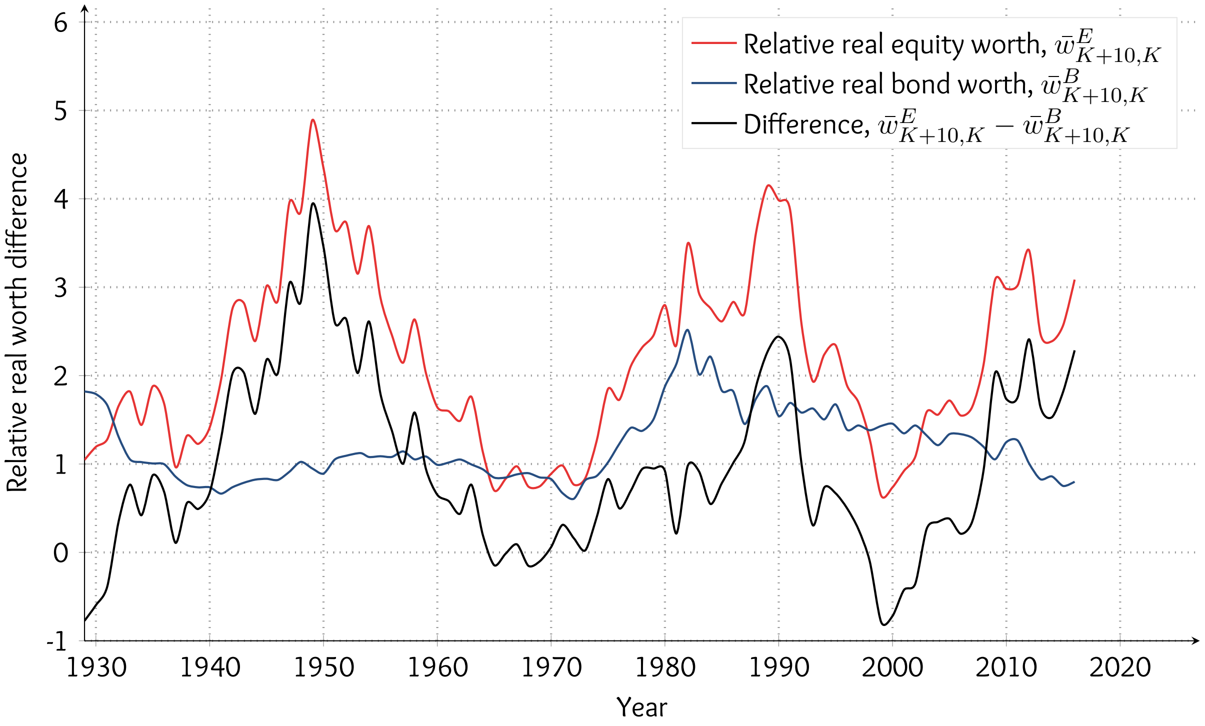

Abstract

A NUMERICAL MODEL IS PRESENTED[1] in an attempt to better understand, and hence navigate, an investment landscape that is always changing. The entities in this landscape could include equities, bonds, preference shares, listed property, real estate, and cash. The focus of this work, however, is on equities and bonds only. Taking the view of an investor, a simple quantitative time-discrete model is presented to help assess the relative attractiveness of equities and bonds. Along the way, two important concepts are introduced to place equities and bonds on an equal comparative footing. They are the future *relative real equity worth* and the future *relative real bond worth*. The model is applied to US equity and US bond data spanning nearly the last 100 years. A novel quantitative strategy is introduced for investing in equities. The strategy obtains from the numerical model presented here and from careful scrutiny of the historical data. The strategy culminates in a ruleset for signalling when to buy, hold or sell equities at any instant. By applying this ruleset at many instants over the last 100 years, I show that the strategy compares favourably with two other widely used—or at least, widely cited—investment strategies. It is encouraging that in an ever-changing investment landscape, the model, together with the novel signalling ruleset, offers mid- to long-term investors an enhanced capability with which to make sensible investment decisions.

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[1] I declare this to be my own work, entirely. In particular, no AI was used in any research, analysis, synthesis, writing, nor typesetting of this work. In short, AI was not recruited at any time in this work. Errors and inaccuracies are therefore proudly my own.

Contents

1 Introduction 4

2 Inflation adjusted pricing 4

2.1 Forward real pricing 5

2.2 Backward real pricing 7

2.3 US inflation rate 8

3 Equities 9

3.1 Receipt of dividends 9

3.2 Reinvesting dividends 14

3.3 Relative real equity worth 15

3.4 Investment period 16

4 Bonds 18

4.1 Primary bond market 18

4.2 Secondary bond market 20

4.3 Coupon bonds 21

4.4 Secondary bond market only 24

5 Equities versus bonds 28

6 Predictive model 30

6.1 Strategy 1—No signalling, so “Buy and Hold” 30

6.2 Strategy 2—Signalling based only on present-day equity expensiveness 30

6.3 Strategy 3—Signalling based on future equity expensiveness 30

6.3.1 Relative real equity worth 31

6.3.2 Estimating the earning rate per share 32

6.3.3 Estimating the dividend payout ratio 32

6.3.4 Estimating the inflation rate 32

6.3.5 Estimating future PE ratios 32

6.4 Strategy 4—Signalling based on present-day and future equity expensiveness 35

6.5 Put it to the test 35

7 Acknowledgments 40

List of Figures

1 History of the US inflation rate, α_K . . . 8

2 Histories of the US S&P 500 nominal share price, E_K, and the real share price, $\bar{E}_{2026.00-k,2026.00}$ 10

3 Histories of the US S&P 500 nominal earning rate per share, e_K, and the real earning rate per share, $\bar{e}_{2026.00-k,2026.00}$ . . . 11

4 Histories of the US S&P 500 nominal dividend rate per share, d_K, and the real dividend rate per share, $\bar{d}_{2026.00-k,2026.00}$ . . . 11

5 History of the US S&P 500 dividend yield rate, d_K/E_K . . . 12

6 History of the US S&P 500 dividend payout ratio, d_K/e_K . . . 12

7 History of the US S&P 500 share price per earning rate ratio (PE ratio), $\rho_K = E_K/e_K$ . . . 13

8 Histories of relative nominal equity worth, $w^E_{K+10,K}$, with reinvestment of dividends and without reinvestment of dividends, after investing for 10 years . . . 15

9 Histories of relative nominal equity worth, $w^E_{K+10,K}$, and relative real equity worth, $\bar{w}^E_{K+10,K}$, after investing for 10 years . . . 17

10 Histories of relative nominal equity worth, $w^E_{K+10,K}$, and relative real equity worth, $\bar{w}^E_{K+10,K}$, after investing for 10 years . . . 18

11 Histories of the relative real equity worth, $\bar{w}^E_{K+10,K}$, with reinvestment of dividends and without reinvestment of dividends, after investing for 10 years . . . 19

12 Histories of relative real equity worth, $\bar{w}^E_{K+k,K}$, after investing for 2 years, 5 years, 10 years, and 15 years, *without reinvesting* dividends . . . 19

13 History of the US 10–year treasury constant maturity interest rate, β_K . . . 23

14 History of the relative real bond worth, $\bar{w}^B_{K+10,K}$, with reinvestment of coupon receipts and without reinvestment of coupon receipts after investing for 10 years . . . 27

15 History of the difference between the relative real equity worth $\bar{w}^E_{K+10,K}$ (*with dividends reinvested*) and the relative real bond worth $\bar{w}^B_{K+10,K}$ (*with coupons reinvested*) after investing for 10 years . . . 29

16 History of the US S&P 500 future PE ratio criterion, $\rho_{\text{min}}(t_{K+10})$ . . . 34

1 Introduction

The agents in an economy include: *corporations* that provide products and services, *consumers* who benefit from these products and services, and *governments* that facilitate effective functioning of the economy though legislation and policy and by laying down communal infrastructure. To fulfil their respective roles in the economy, corporations and governments require capital. They sometimes choose to obtain this capital by issuing corporate and government *bonds*. For an investor, the purchase of such bond issuances is considered a type of investment. Another important type of investment is *equities*. By purchasing part ownership of a corporation, an investor obtains the right to share in any profits generated by the corporation. Other types of investments exist, such as cash, preference shares, listed property, and real estate. They all involve one or more of the abovementioned agents in the economy. It is therefore reasonable to expect them to be affected, at least in part, by fluctuations in the state of the economy. However, they will not be considered here. Here, the focus is on bonds and equities.

This work, then, presents a quantitative analysis of the relative attractiveness of equities and bonds as investment instruments, taking the view of an investor. A simple quantitative time-discrete model is introduced to help assess their relative attractiveness as investment instruments. In the model, the time intervals between which investment decisions are taken are assumed to be large, at least a few months, and the parameters influencing such decisions are assumed to be macroeconomic is nature. The model draws on US equity and US bond data spanning nearly the last 100 years.

Unfortunately, any quantitative economic modelling work must account for inflation—that insidious and apparently inevitable erosion of the purchasing power of money. In Section 2, the notions of *forward real pricing* and *backward real pricing* are formally introduced to account for inflation. They are then incorporated into the ensuing modelling work.

A number of concepts pertaining to an investor's investment worth are introduced in this work. Of these, two stand out as especially important because they enable placing equities and bonds on an equal comparative footing. They are the future *relative real equity worth* and the future *relative real bond worth*. As the story unfolds here, I hope the reader might come to appreciate their importance as I did. An investor's potential worth derived from equity investments is detailed in Section 3 on page 9, where the investor could have been situated at any moment in the last 100 years or so. Similarly, an investor's potential worth from investing in bonds is contemplated in Section 4 on page 18. In Section 5 on page 28, investing in equities is compared quantitatively to investing in bonds.

Finally, following the results in Section 5, Section 6 on page 30 focusses specifically on equities. A predictive model is developed, resulting in estimates of lower bounds on the future expensiveness of equities. The predictive model builds on six key insights that were obtained by careful scrutiny of the historical time series data. These insights are itemised on page 31. A novel quantitative strategy is introduced for investing in equities. The strategy obtains from the numerical model presented here and from careful scrutiny of the historical data. The strategy culminates in a ruleset for signalling when to buy, hold or sell equities at any moment. By applying this ruleset at moments over the last 100 years, I show that the strategy compares favourably with two other widely used—or at least, widely cited—investment strategies. The model presented here, together with the novel signalling ruleset, offers mid- to long-term investors an enhanced capability with which to make sensible investment decisions.

2 Inflation adjusted pricing

Inflation obscures the intrinsic value of products and services available in an economy. This is because in an unregulated economy, prices of goods and services are allowed to vary over time. Therefore, if we wish to better understand the actions of agents in an economy, the effects of inflation must be accounted for. Doing so impels a distinction between the notions of *real pricing* and *nominal pricing*. In this article, I derive formulae for *forward real pricing* and *backward real pricing* over time periods in which inflation and savings may vary. Along the way, I derive the well known Fisher Equation from first principles. In financial mathematics, the Fisher Equation expresses the relationship between nominal interest rates, real interest rates, and inflation.

In the ensuing analysis here, I assume that time flows discretely rather than continuously. I do so because when central state statisticians release values for the inflation rate, they assume discrete time.

2.1 Forward real pricing

Purchasing ability. I begin by stating the obvious. To say that at time t_J we have a monetary amount P_J means that at t_J we have the ability to exchange P_J units of currency for N_J products or services, say. This is because, taken together, a supplier of those products or services has deemed them to be of equal value to P_J. Stated reciprocally, at t_J we are able to exchange a single unit of our currency for N_J/P_J units of product or service.

But suppose that instead of making the exchange of currency for products or services at t_J, we place P_J in an investment that offers a *nominal return* over the $(t_J, t_{J+1}]$ time interval such that when we arrive at time t_{J+1}, we have a nominal monetary amount P_{J+1}. Unfortunately for us as holders of P_{J+1}, the supplier has now deemed that at t_{J+1} the N_J products or services are no longer worth P_J, but rather $(1+\alpha_{J+1}\Delta t_{J+1})P_J$, where α_{J+1} is the *time rate of relative increase in the monetary value* of the N_J items at t_{J+1}, i.e., the inflation rate, and where

$$\Delta t_{J+1} \equiv t_{J+1} - t_J$$

So at t_{J+1}, to purchase the N_J items from the supplier, we would have to dispense with $(1+\alpha_{J+1}\Delta t_{J+1})P_J$ units of our currency. Or stated reciprocally, at t_{J+1}, a single unit of our currency would now be able to purchase only $N_J/[(1+\alpha_{J+1}\Delta t_{J+1})P_J]$ items. But since at t_{J+1} we now have P_{J+1} units of currency, we are in fact able to purchase

$$N_{J+1} = \frac{N_J}{(1+\alpha_{J+1}\Delta t_{J+1})P_J} P_{J+1} \tag{1}$$

units of product or service.

It is therefore clear that the *real relative return* of the investment over the $(t_J, t_{J+1}]$ time interval is not the relative change in our nominal monetary value. Instead, it must be the *relative change in our purchasing ability*, namely:

$$\frac{N_{J+1} - N_J}{N_J} \equiv \bar{\sigma}_{J+1,J}\Delta t_{J+1} = \frac{1}{1+\alpha_{J+1}\Delta t_{J+1}} \frac{P_{J+1}}{P_J} - 1 \tag{2}$$

where $\bar{\sigma}_{J+1,J}$ is defined here as the *time rate of relative change in purchasing ability* over the $(t_J, t_{J+1}]$ time interval.

Fisher Equation. Equation (2) may be used to derive the well known Fisher Equation. If the time rate of nominal relative change in our P_J monetary amount is known to be σ_{J+1} over $(t_J, t_{J+1}]$, and is assumed constant over the interval, then by definition of σ we may write

$$P_{J+1} = (1+\sigma_{J+1}\Delta t_{J+1})P_J$$

In many investment instruments, such as, moneymarket accounts, bank savings accounts, and bond investments, a value for σ is well known. Substituting into (2) gives

$$\bar{\sigma}_{J+1,J}\Delta t_{J+1} = \frac{1+\sigma_{J+1}\Delta t_{J+1}}{1+\alpha_{J+1}\Delta t_{J+1}} - 1 \tag{3}$$

If $\alpha_{J+1}\Delta t_{J+1}$ is small, we may apply a first-order Taylor series approximation to the denominator, yielding a form of the Fisher Equation:

$$\begin{aligned}
\bar{\sigma}_{J+1,J}\Delta t_{J+1} &= (1+\sigma_{J+1}\Delta t_{J+1})\,(1+\alpha_{J+1}\Delta t_{J+1})^{-1} - 1 \\
&\approx (1+\sigma_{J+1}\Delta t_{J+1})\,(1-\alpha_{J+1}\Delta t_{J+1}) - 1 \\
&\approx (\sigma_{J+1} - \alpha_{J+1})\Delta t_{J+1}
\end{aligned}$$

Real price. The real relative change in our purchasing ability (Eq. (2)) over the $(t_J, t_{J+1}]$ time interval admits the notion of a *real price*, $\bar{P}_{i,j}$, applicable at some time t_i, and relative to pricing at some other time t_j. The real price is a fictitious one. It is merely an adjustment to the nominal price of an item to offset the effect of inflation on pricing. To make the adjustment, we simply stipulate that the *relative change in the real price over* $(t_J, t_{J+1}]$ *matches the relative change in our purchasing ability*. That is, we use (2) and stipulate that

$$\begin{aligned} \bar{P}_{J,J} &= P_J \\ \frac{\bar{P}_{J+1,J} - \bar{P}_{J,J}}{\bar{P}_{J,J}} &= \frac{N_{J+1} - N_J}{N_J} = \bar{\sigma}_{J+1,J} \Delta t_{J+1} \end{aligned} \tag{4}$$

The condition $\bar{P}_{J,J} = P_J$ conveys that at some specified initial time t_J, we wish for the real price to match the nominal price. Note that throughout this analysis, a bar over a symbol denotes a real quantity instead of a nominal one. The stipulation implies that

$$\begin{aligned} \bar{P}_{J,J} &= P_J \\ \bar{P}_{J+1,J} &= (1 + \bar{\sigma}_{J+1,J} \Delta t_{J+1}) \bar{P}_{J,J} = \left(\frac{1 + \sigma_{J+1} \Delta t_{J+1}}{1 + \alpha_{J+1} \Delta t_{J+1}} \right) \bar{P}_{J,J} \qquad \text{(using (3))} \\ &= \frac{P_{J+1}}{1 + \alpha_{J+1} \Delta t_{J+1}} \qquad \text{(using (2))} \\ &= \frac{1}{1 + \alpha_{J+1} \Delta t_{J+1}} \frac{P_{J+1}}{P_J} \bar{P}_{J,J} \end{aligned} \tag{5}$$

Next, consider a subsequent time t_{J+2}. At t_{J+2}, our nominal monetary amount is P_{J+2}, say. Again, unfortunately for us as holders of P_{J+2}, the supplier has deemed that the N_J products or services are no longer worth P_J, but instead are worth $(1 + \alpha_{J+2} \Delta t_{J+2}) P_{J+1}$, where α_{J+2} is the inflation rate at t_{J+2}. So to purchase the N_J items from the supplier at t_{J+2}, we would have to dispense with $(1 + \alpha_{J+2} \Delta t_{J+2})(1 + \alpha_{J+1} \Delta t_{J+1}) P_J$ units of our currency. And stated reciprocally, at t_{J+2} a single unit of our currency would now be able to purchase only $N_J / [(1 + \alpha_{J+2} \Delta t_{J+2})(1 + \alpha_{J+1} \Delta t_{J+1}) P_J]$ items. But since we now have P_{J+2} units of currency, we are in fact able to purchase

$$N_{J+2} = \frac{N_J}{(1 + \alpha_{J+1} \Delta t_{J+1})(1 + \alpha_{J+2} \Delta t_{J+2}) P_J} P_{J+2} \tag{6}$$

such items of product or service.

To obtain a sensible real price at t_{J+2}, we once again stipulate that the relative change of the real price over the $(t_J, t_{J+2}]$ time interval matches the relative change of our purchasing ability over that same interval:

$$\begin{aligned} \bar{P}_{J,J} &= P_J \\ \frac{\bar{P}_{J+2,J} - \bar{P}_{J,J}}{\bar{P}_{J,J}} &= \frac{N_{J+2} - N_J}{N_J} \end{aligned}$$

From this we obtain, using (6)

$$\begin{aligned} \bar{P}_{J+2,J} &= \frac{P_{J+2}}{(1 + \alpha_{J+1} \Delta t_{J+1})(1 + \alpha_{J+2} \Delta t_{J+2})} \\ &= \frac{1}{1 + \alpha_{J+2} \Delta t_{J+2}} \frac{P_{J+2}}{P_{J+1}} \bar{P}_{J+1,J} \end{aligned} \tag{7}$$

Following similar reasoning, it is easy to show that a sensible real price at yet a subsequent time t_{J+3}, relative to pricing at t_J, is

$$\begin{aligned} \bar{P}_{J+3,J} &= \frac{P_{J+3}}{(1 + \alpha_{J+1} \Delta t_{J+1})(1 + \alpha_{J+2} \Delta t_{J+2})(1 + \alpha_{J+3} \Delta t_{J+3})} \\ &= \frac{1}{1 + \alpha_{J+3} \Delta t_{J+3}} \frac{P_{J+3}}{P_{J+2}} \bar{P}_{J+2,J} \end{aligned} \tag{8}$$

By extension of (5), (7) and (8) to the $(t_J, t_{J+k}]$ time interval, a sensible future real price at t_{J+k}, relative to pricing at t_J, is

$$\boxed{\begin{aligned} \bar{P}_{J+k,J} &= \frac{P_{J+k}}{\prod_{l=1}^{k}(1+\alpha_{J+l}\Delta t_{J+l})} \\ &= \frac{1}{1+\alpha_{J+k}\Delta t_{J+k}}\frac{P_{J+k}}{P_{J+k-1}}\bar{P}_{J+k-1,J} \\ &\qquad \text{for } k=1,2,\dots, \text{ with } \bar{P}_{J,J}=P_J \end{aligned}} \tag{9}$$

where $\Delta t_{J+l} \equiv t_{J+l} - t_{J+l-1}$. The second form of this specification of $\bar{P}_{J+k,J}$ is a recursive implicit one in that the real price at time t_{J+k} depends on the real price at time t_{J+k-1}, namely, $\bar{P}_{J+k-1,J}$. So if $\bar{P}_{J+k-1,J}$ is known, then the second form offers a computational advantage over the first form because the second form requires fewer floating point calculations to obtain $\bar{P}_{J+k,J}$.

2.2 Backward real pricing

Alternatively, suppose we wish to calculate historical real prices relative to a current or future price. Then we implicitly agree that the real price must match the nominal price at a current or future time t_J, for $J > 0$. Nominal prices at times earlier than t_J will then be adjusted for inflation to admit real prices relative to the price at t_J.

Setting J to $J-1$ in (1) and rearranging gives

$$N_{J-1} = (1+\alpha_J\Delta t_J)\frac{P_{J-1}}{P_J}N_J \tag{10}$$

Whereas criterion (4) was used to calculate forward real prices $\bar{P}_{J+k,J}$ in (9), to calculate a backward real price $\bar{P}_{J-1,J}$ at time t_{J-1}, we now stipulate that

$$\bar{P}_{J,J} = P_J$$
$$\frac{\bar{P}_{J-1,J} - \bar{P}_{J,J}}{\bar{P}_{J,J}} = \frac{N_{J-1} - N_J}{N_J}$$

Using (10) gives

$$\bar{P}_{J-1,J} = (1+\alpha_J\Delta t_J)P_{J-1} \tag{11}$$

Setting J to $J-2$ in (1) and rearranging gives

$$\begin{aligned} N_{J-2} &= (1+\alpha_{J-1}\Delta t_{J-1})\frac{P_{J-2}}{P_{J-1}}N_{J-1} \\ &= (1+\alpha_{J-1}\Delta t_{J-1})(1+\alpha_J\Delta t_J)\frac{P_{J-2}}{P_J}N_J \end{aligned} \tag{12}$$

To calculate a backward real price $\bar{P}_{J-2,J}$ at time t_{J-2}, we stipulate that

$$\bar{P}_{J,J} = P_J$$
$$\frac{\bar{P}_{J-2,J} - \bar{P}_{J,J}}{\bar{P}_{J,J}} = \frac{N_{J-2} - N_J}{N_J}$$

Using (12) gives

$$\bar{P}_{J-2,J} = (1+\alpha_J\Delta t_J)(1+\alpha_{J-1}\Delta t_{J-1})P_{J-2}$$

And also, using (11)

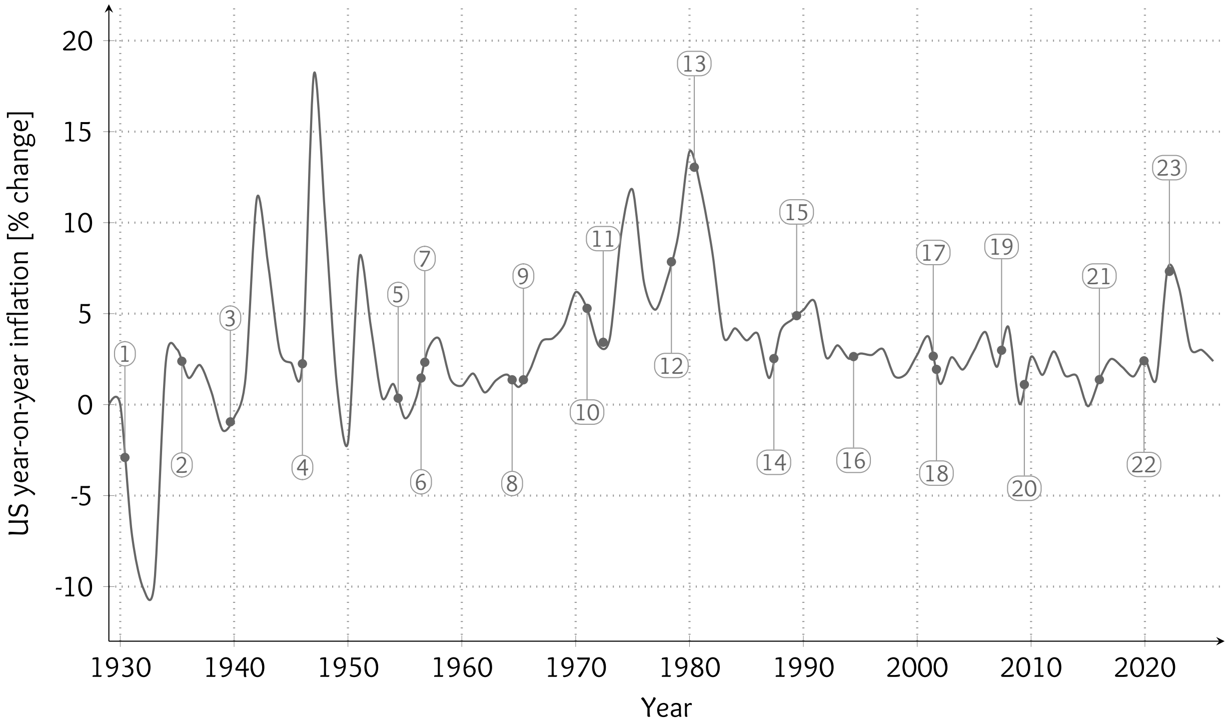
$$\bar{P}_{J-2,J} = (1+\alpha_{J-1}\Delta t_{J-1})\frac{P_{J-2}}{P_{J-1}}\bar{P}_{J-1,J}$$

By extension to the $(t_{J-k}, t_J]$ time interval, a sensible backward real price at t_{J-k}, relative to pricing at t_J, is

$$\boxed{\begin{aligned} \bar{P}_{J-k,J} &= \prod_{l=0}^{k-1} \left(1 + \alpha_{J-l}\Delta t_{J-l}\right) P_{J-k} \\ &= \left(1 + \alpha_{J-k+1}\Delta t_{J-k+1}\right) \frac{P_{J-k}}{P_{J-k+1}} \bar{P}_{J-k+1,J} \\ &\qquad\qquad\qquad \text{for } k = 1, 2, \ldots, J, \text{ with } \bar{P}_{J,J} = P_J \end{aligned}} \tag{13}$$

2.3 US inflation rate

The US inflation rate over the last 100 years or so is shown in Figure 1 on page 8.



$\boxed{1}$ Great Depression begins. $\boxed{2}$ Social Security Bill passed. $\boxed{3}$ Spending for World War II begins. $\boxed{4}$ 1945–1973: Post-World War II prosperity. $\boxed{5}$ 34.8% long-term peak in union membership. $\boxed{6}$ Unemployment Assistance Act passed. $\boxed{7}$ Work on Interstate Highway System begins. $\boxed{8}$ \$12bn tax cut. $\boxed{9}$ Medicare, Medicaid enacted. $\boxed{10}$ Nixon's wage & price controls. $\boxed{11}$ Arab oil embargo during Arab-Israeli war. $\boxed{12}$ Airline Deregulation Act passed. $\boxed{13}$ Inflation reaches 13.5%. $\boxed{14}$ U.S. stock market crash. $\boxed{15}$ Savings and loan institutions fail. $\boxed{16}$ NAFTA launched to remove trade tariffs. $\boxed{17}$ Bush's \$350bn tax cut passed. $\boxed{18}$ September 11, 2001 attacks. $\boxed{19}$ Subprime mortgage crisis. $\boxed{20}$ \$11tn national debt, \$1.4tn deficit. $\boxed{21}$ \$20tn national debt=107% GDP. $\boxed{22}$ COVID-19 pandemic. $\boxed{23}$ Russia invades Ukraine.

Figure 1: History of the US inflation rate, α_K, based on the Consumer Price Index for all urban consumers. The long-term time averaged inflation rate is 3.15% per year. (Raw data sources: Federal Reserve Bank of St Louis,[1] Yale University.[2])

3 Equities

BY INVESTING IN EQUITIES, an investor hopes to increase redeemable monetary worth in two ways. Firstly, the share of equities owned by the investor might be sold at a price higher than what it was bought at. Secondly, by owning the equity share, the investor may obtain a proportional share in dividends that are distributed from time to time. The investor may elect simply to receive the proceeds of the dividend distribution, or to use the proceeds to purchase additional equities, thereby increasing his/her share.

3.1 Receipt of dividends

Suppose that at some time t_K we have an amount W_K to invest. If we choose to invest only in an equity instrument, then at some subsequent time t_{K+1}, the investment must be worth, nominally,

$$W^E_{K+1} = NE_{K+1} + ND_{K+1}$$

where N is the number of shares purchased at t_K, E_{K+1} is the share price at time t_{K+1}, and D_{K+1} is the dividend per share received over the $(t_K, t_{K+1}]$ time interval. If d_u is the recorded time rate of receipt of nominal[2] dividends per share at time t_u, then

$$W^E_{K+1} = NE_{K+1} + Nd_{K+1}\Delta t_{K+1}$$

After two subsequent periods, the investment must be worth

$$W^E_{K+2} = N\left[E_{K+2} + d_{K+1}\Delta t_{K+1} + d_{K+2}\Delta t_{K+2}\right]$$

And so on for the third and subsequent periods. Therefore, at any time t_{K+k}

$$W^E_{K+k} = N\left[E_{K+k} + \sum_{i=1}^{k} d_{K+i}\Delta t_{K+i}\right]$$

Since the amount available to invest at time t_K is $W_K = NE_K$, the investor's *relative nominal equity worth* at t_{K+k} relative to that at t_K is

$$\boxed{w^E_{K+k,K} = \frac{W^E_{K+k}}{W_K} = \frac{1}{E_K}\left[E_{K+k} + \sum_{i=1}^{k} d_{K+i}\Delta t_{K+i}\right]} \tag{14}$$

If the time intervals all happen to be of equal size, then $\Delta t_{K+i} = \Delta t$ for all i, so that

$$w^E_{K+k,K} = \frac{W^E_{K+k}}{W_K} = \frac{1}{E_K}\left[E_{K+k} + \Delta t\sum_{i=1}^{k} d_{K+i}\right]$$

We now assume that the investor invests in an equity instrument such as an exchange-traded fund (ETF) whose price per share, E, closely follows the price per share of the Standard and Poor's 500 stock market index, or simply the S&P 500 index.[3] The price of the index is a weighted sum of the prices of 500 publicly listed US-based companies. The weighting is done by the companies' market capitalisation. About 80% of the total market capitalisation of publicly listed companies in the US contributes to the index. A history of the index's nominal price and a real price is shown in Figure 2. The real price history was calculated using (9) on page 7 and (13) on page 8.

The histories of the rate of nominal earning per share and real earning per share of the S&P 500 index over the same period is shown in Figure 3 on page 11. The history of the time rate of distribution of dividends per share of the index over the same period is shown in Figure 4 on page 11. The history of the dividend yield rate of the index is shown in Figure 5 on page 12. The *dividend payout ratio* is the proportion of earnings

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[2] The 'nominal' adjective refers to an actual amount, i.e., an amount *not* adjusted for inflation.

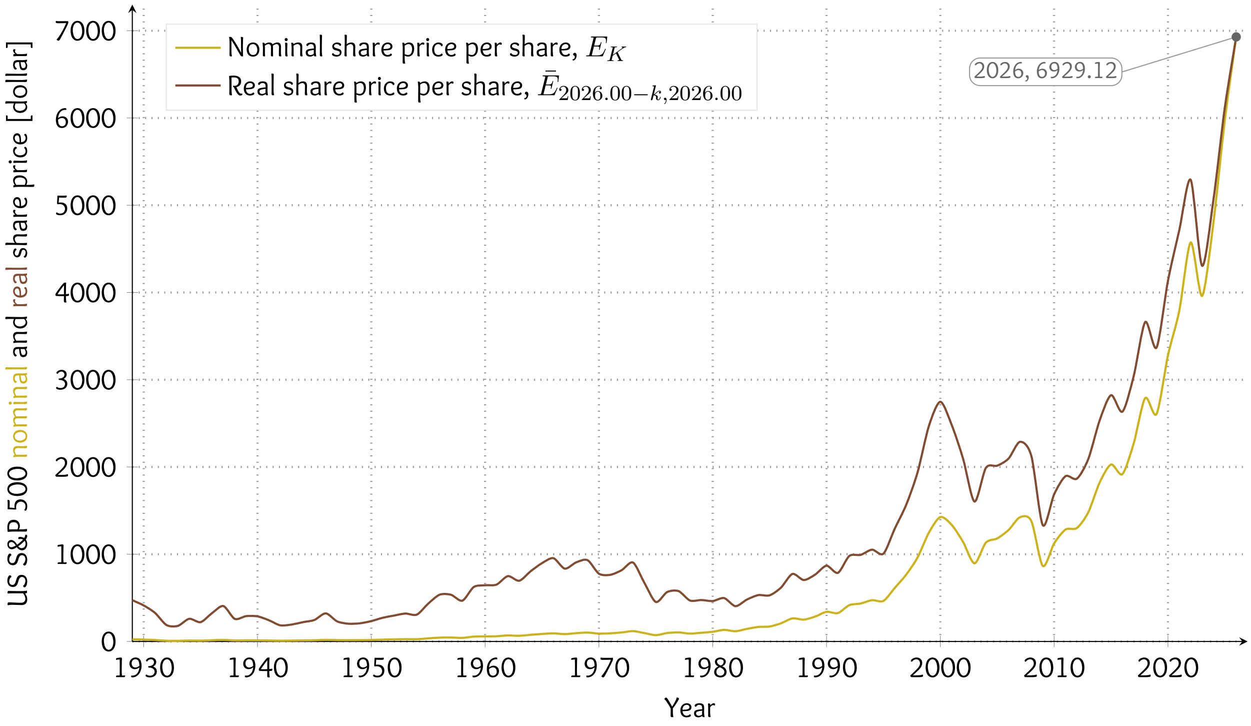

Figure 2: Histories of the US S&P 500 nominal (i.e., inflation unadjusted) share price, E_K, and the real (i.e., inflation adjusted) share price, $\bar{E}_{2026.00-k,2026.00}$, relative to the value in 2026. (Raw data sources: `www.multpl.com`,[4] Federal Reserve Bank of St Louis,[1] Yale University.[2])

paid to a company's shareholders as dividends. The history of the dividend payout ratio of the S&P 500 index fund is shown in Figure 6 on page 12.

An important measure of the expensiveness of an equity is the well known *PE Ratio*, ρ. Or stated in full, the PE Ratio at any time t_K is the ratio of the equity's nominal share price to the equity's rate of nominal earning:

$$\rho_K \equiv E_K / e_K \qquad \text{at some time } t_K$$

The history of the PE Ratio of the S&P 500 index is shown below in Figure 7 on page 13.

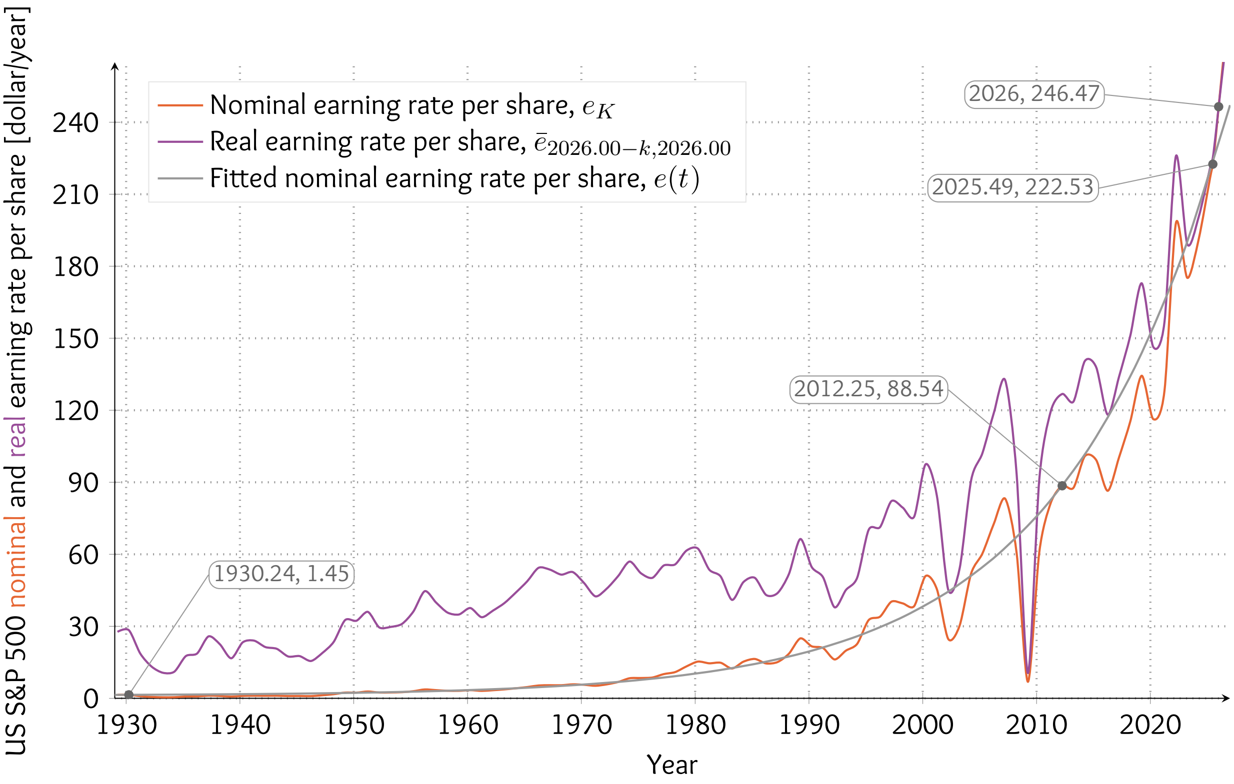

Figure 3: Histories of the US S&P 500 nominal (i.e., inflation unadjusted) earning rate per share, e_K, and the real (i.e., inflation adjusted) earning rate per share, $\bar{e}_{2026.00-k,2026.00}$, relative to the value in 2026. (Raw data sources: S&P Global,[5] Yale University[6], Macrotrends,[7] Guru Focus,[8] Federal Reserve Bank of St Louis.[1])

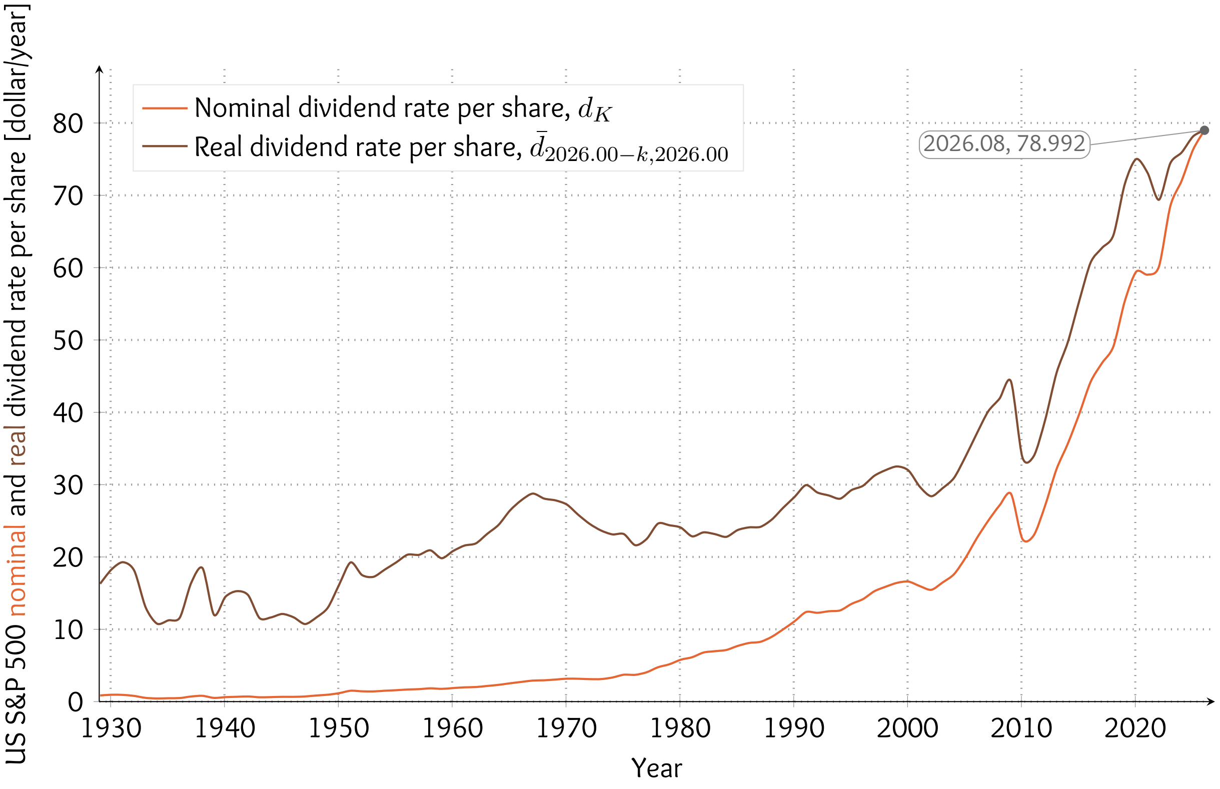

Figure 4: Histories of the US S&P 500 nominal dividend rate per share, d_K, and the real dividend rate per share, $\bar{d}_{2026.00-k,2026.00}$, relative to the value in 2026. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University.[2])

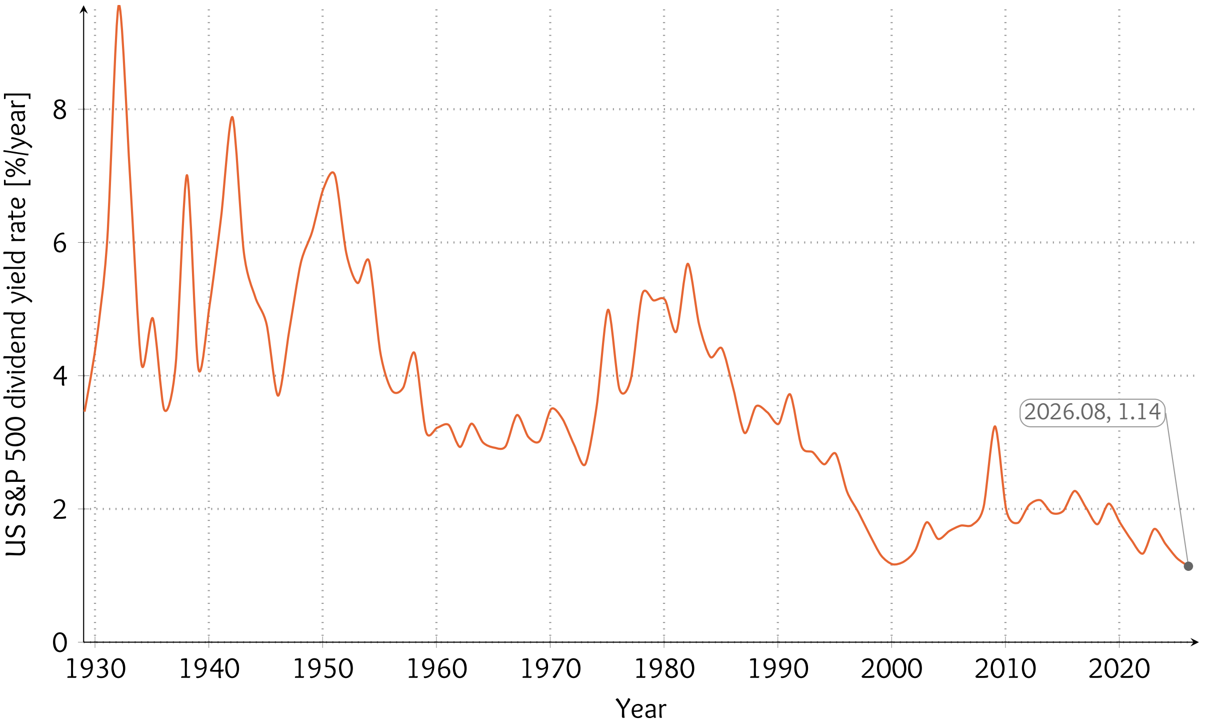

Figure 5: History of the US S&P 500 dividend yield rate, d_K/E_K. (Raw data sources: `www.multpl.com`.[4, 9])

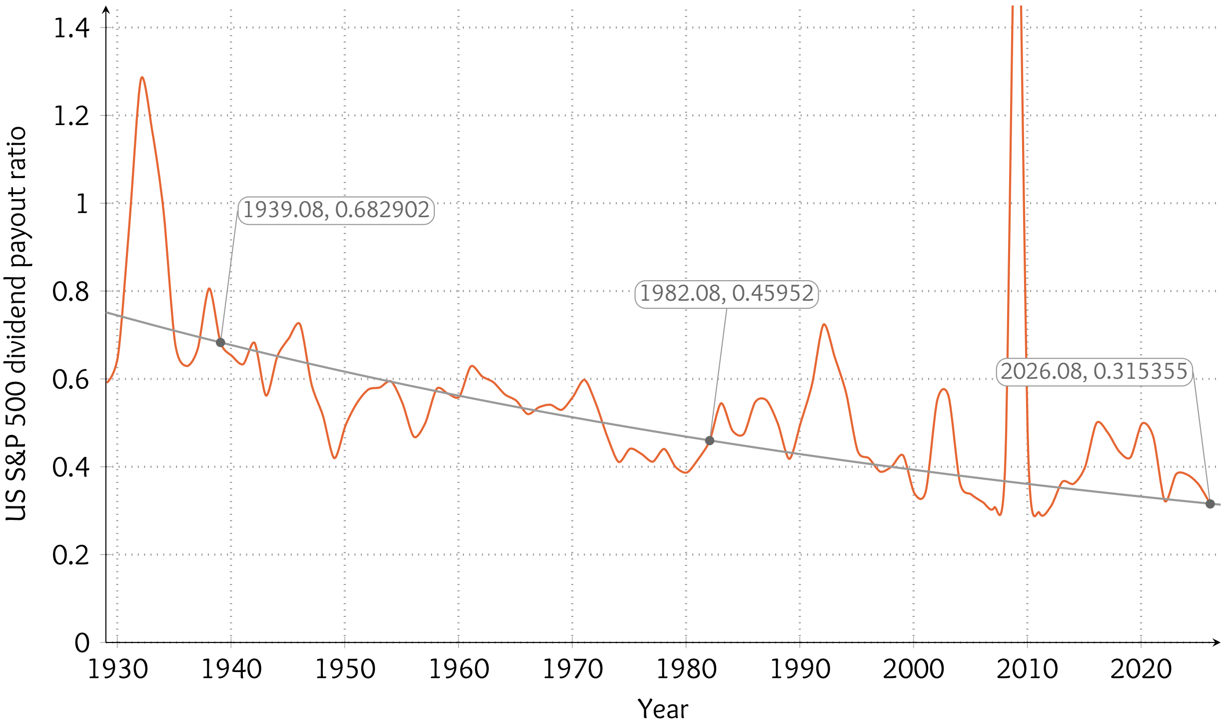

Figure 6: History of the US S&P 500 dividend payout ratio, d_K/e_K. (Raw data sources: `www.multpl.com`,[4, 9] S&P Global,[5] Yale University.[6])

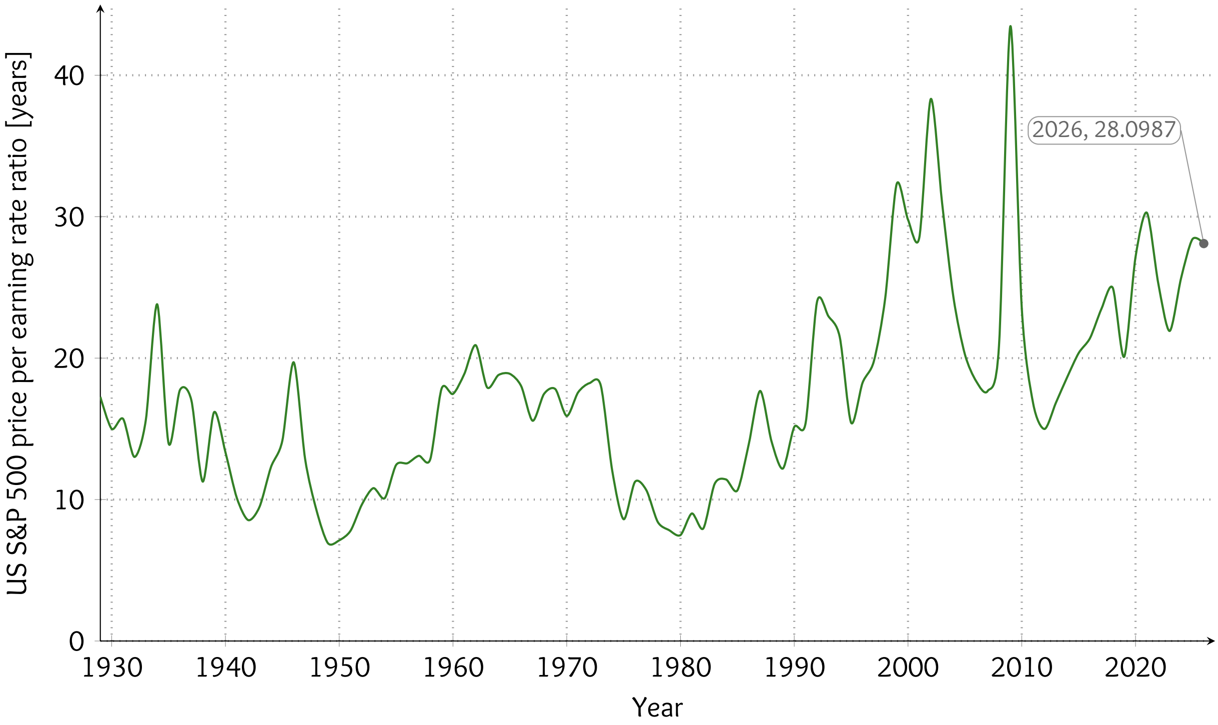

Figure 7: History of the US S&P 500 share price per earning rate ratio (PE ratio), $\rho_K = E_K/e_K$. The long-term time average of the ratio is about 17.5 years. (Raw data sources: `www.multpl.com`,[4] Federal Reserve Bank of St Louis,[1] Yale University,[2] S&P Global,[5] Macrotrends,[7] Guru Focus,[8] Yale University.[6])

3.2 Reinvesting dividends

Instead of simply receiving the proceeds of a dividend distribution, the equity investor may choose to use the proceeds to purchase additional equities.

Suppose, as before, that at some time t_K we have an amount W_K to invest. If we choose to invest only in an equity instrument, then at some subsequent time t_{K+1}, the dividend amount ND_{K+1} is now used to purchase ND_{K+1}/E_{K+1} additional shares. Recall from page 9 that d_{K+1} is the recorded time rate of receipt of nominal dividends per share at time t_{K+1}. After the purchase, the investment must be worth

$$\begin{aligned} W^E_{K+1} &= N_{K+1}E_{K+1} \\ &= \left(N + \frac{ND_{K+1}}{E_{K+1}}\right) E_{K+1} \\ &= N\left(1 + \frac{d_{K+1}\Delta t_{K+1}}{E_{K+1}}\right) E_{K+1} \end{aligned}$$

Similarly, at some subsequent time t_{K+2}, $N_{K+1}D_{K+2}/E_{K+2}$ additional shares are purchased, so that

$$\begin{aligned} W^E_{K+2} &= N_{K+2}E_{K+2} \\ &= \left(N_{K+1} + \frac{N_{K+1}D_{K+2}}{E_{K+2}}\right) E_{K+2} \\ &= N_{K+1}\left(1 + \frac{D_{K+2}}{E_{K+2}}\right) E_{K+2} \\ &= N\left(1 + \frac{D_{K+1}}{E_{K+1}}\right)\left(1 + \frac{D_{K+2}}{E_{K+2}}\right) E_{K+2} \\ &= N\left(1 + \frac{d_{K+1}\Delta t_{K+1}}{E_{K+1}}\right)\left(1 + \frac{d_{K+2}\Delta t_{K+2}}{E_{K+2}}\right) E_{K+2} \end{aligned}$$

And so on for the third and subsequent periods, giving the investor's relative nominal equity worth at time t_{K+k} as

$$\boxed{w^E_{K+k,K} = \prod_{i=1}^{k}\left(1 + \frac{d_{K+i}\Delta t_{K+i}}{E_{K+i}}\right)\frac{E_{K+k}}{E_K} \qquad \text{for any } k > 0} \tag{15}$$

If the time intervals are all of equal size, then

$$w^E_{K+k,K} = \prod_{i=1}^{k}\left(1 + \frac{d_{K+i}}{E_{K+i}}\,\Delta t\right)\frac{E_{K+k}}{E_K} \qquad \text{for any } k > 0 \tag{16}$$

The histories in Figure 2 on page 10 and Figure 4 on page 11 were used to compute the histories of the relative nominal equity worth, $w^E_{K+10,K}$, shown below in Figure 8 on page 15. What is Figure 8 reporting? An equity investor who invested for a period of 10 years starting in 1970 would have obtained a relative nominal equity worth $w^E_{1970+10,1970}$ equal to 1.9 or 1.7 depending, respectively, on whether or not dividends were reinvested. In other words, nominally, the investment would have increased by a factor 1.9 or 1.7 over 10 years, beginning 1970.

Conventional wisdom dictates that a prudent equity investor should reinvest dividends instead of taking receipt of them. A comparison of the two histories of $w^E_{K+10,K}$ shown in Figure 8 supports this wisdom. The long-term history obtained without reinvesting dividends was computed using (14) on page 9, and the history obtained by reinvesting dividends was computed using (15). In both, the period of equity investment was set to 10 years. The apparent near coincidence of the two histories starting early 1990s reflects the dramatic decline in dividend yield rates from that time till the present, as shown in Figure 5 on page 12. Nevertheless, it is significant that at no time in the two histories would the investor's relative worth have been higher if dividends were not reinvested.

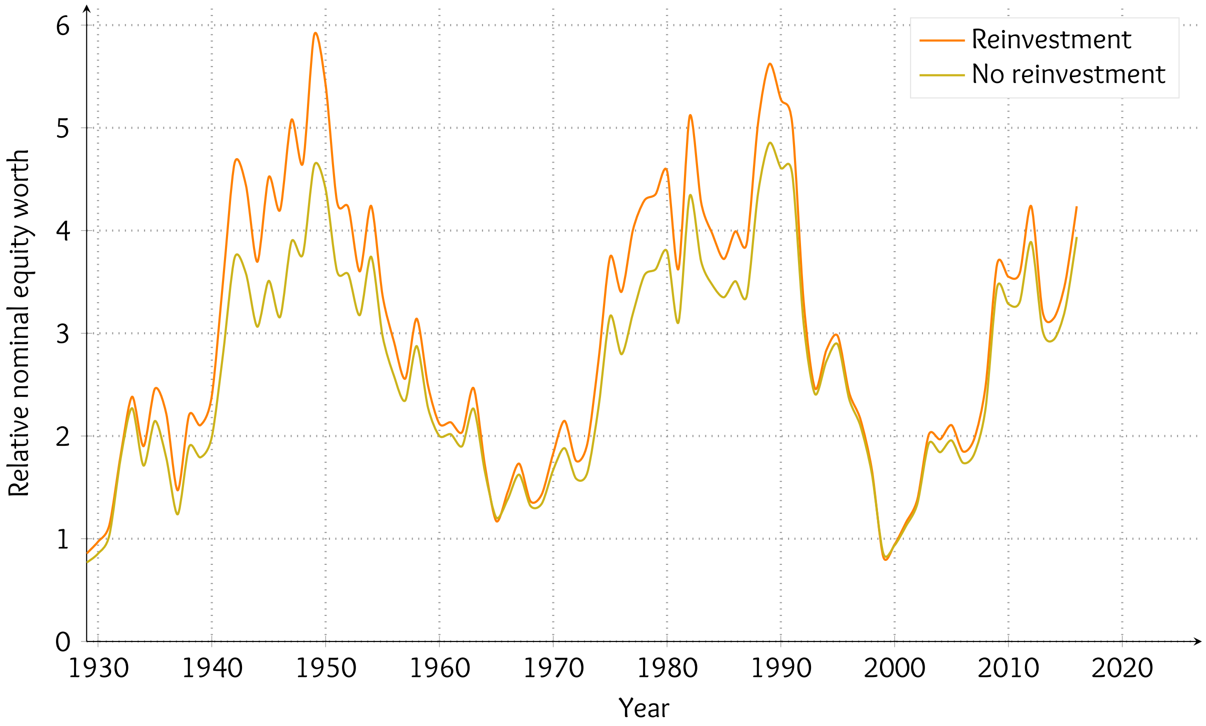

Figure 8: Histories of relative nominal equity worth, $w^E_{K+10,K}$, with reinvestment of dividends and without reinvestment of dividends, after investing for 10 years ($k = 10$). The time average of the worth with reinvestment of dividends is about 3.01, and without reinvestment of dividends is about 2.64. The time averaged arithmetic difference between them is about 0.37. (Raw data sources: `www.multpl.com`.[4, 9])

3.3 Relative real equity worth

A superficial glance at either of the histories in Figure 8 might impute some optimism in that a 10 year investment period in equities almost always produces a relative equity worth exceeding unity. In other words, the investor's nominal worth almost never declines over a 10 year investment period. However, that initial optimism is partly misplaced because it neglects the eroding effect of inflation. Inflation works to erode an investor's purchasing ability over time even though the investor's nominal worth may increase over time. We must obviously account for this erosion of purchasing ability.

So, stationed at t_K, we focus once again on an investment worth at t_{K+k}, but which is now based on pricing at t_J, for some $J > 0$. That is, we must calculate the *real equity investment worth* at t_{K+k}, using (9), as

$$\bar{W}^E_{K+k,J} = \bar{W}^E_{J+(K-J+k),J} = \frac{W^E_{J+(K-J+k)}}{\displaystyle\prod_{l=1}^{K-J+k} (1 + \alpha_{J+l}\Delta t_{J+l})}$$

Doing so, the investor's *relative real equity worth* at t_{K+k} based on pricing at t_J is then

$$\bar{w}^E_{K+k,J} \equiv \frac{\bar{W}^E_{K+k,J}}{\bar{W}^E_{K,J}} = \frac{\bar{W}^E_{J+(K-J+k),J}}{\bar{W}^E_{J+(K-J),J}} = \frac{W^E_{J+(K-J+k)}}{W^E_{J+(K-J)}} \cdot \frac{\displaystyle\prod_{m=1}^{K-J} (1 + \alpha_{J+m}\Delta t_{J+m})}{\displaystyle\prod_{l=1}^{K-J+k} (1 + \alpha_{J+l}\Delta t_{J+l})}$$

But

$$\begin{aligned}\prod_{l=1}^{K-J+k}(1+\alpha_{J+l}\Delta t_{J+l}) &= \prod_{m=1}^{K-J}(1+\alpha_{J+m}\Delta t_{J+m})\prod_{l=K-J+1}^{K-J+k}(1+\alpha_{J+l}\Delta t_{J+l})\\ &= \prod_{m=1}^{K-J}(1+\alpha_{J+m}\Delta t_{J+m})\prod_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})\end{aligned}$$

So that

$$\boxed{\begin{aligned}\bar{w}^E_{K+k,J} \equiv \frac{\bar{W}^E_{K+k,J}}{\bar{W}^E_{K,J}} &= \frac{W^E_{J+(K-J+k)}}{W^E_{J+(K-J)}}\cdot\frac{1}{\prod_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})}\\ &= \frac{1}{\prod_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})}\cdot\frac{W^E_{K+k}}{W^E_K}\\ &= \frac{w^E_{K+k,K}}{\prod_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})}\\ &= \bar{w}^E_{K+k,K}\end{aligned}} \tag{17}$$

We conclude that the relative real equity worth at some time t_{K+k} relative to that at t_K and based on pricing at some other time t_J is sensitive to the eroding effects of inflation only over the $(t_K, t_{K+k}]$ time interval, and not over the $(t_J, t_{K+k}]$ interval. Therefore, in what follows, we work with $\bar{w}^E_{K+k,K}$ rather than with $\bar{w}^E_{K+k,J}$.

A long-term history of the relative nominal equity worth $w^E_{K+10,K}$ obtained without reinvesting dividends (i.e., Eq. (14)), and that of the corresponding relative real equity worth $\bar{w}^E_{K+10,K}$ in (17), is shown in Figure 9 for an investment period of 10 years ($k = 10$). The nominal history curve shown in Figure 9 is obviously a repeat of the "No reinvestment" curve in Figure 8 on page 15.

A comparison of the real and nominal curves in Figure 9 dampens the abovementioned optimism somewhat. Unsurprisingly, the real curve consistently lies below the nominal curve, lower by an average 0.72 over the long-term. If instead, dividends are reinvested over a 10 year period, the finding is similar, as shown in Figure 10 on page 18, except that the average difference between the nominal and real curves is then about 0.83.

Still, optimism indeed remains because, with the exception of about a one-year interval around 1937, a decade centred on 1969, and about a three-year interval centred on 2000, an investor's relative real equity worth remained above unity. In other words, the investor's purchasing ability arising from his/her 10 year equity investment would have declined only in those three periods. And, this is irrespective of whether or not dividends were reinvested.

For completeness, following the nominal history results in Figure 8 on page 15, corresponding real histories for $\bar{w}^E_{K+10,K}$ (Eq. (17)) are shown in Figure 11 on page 19, again with $k = 10$. The time averaged difference between the two histories reveal that reinvesting dividends boosts an investor's relative real worth by 26% on average over a 10 year period.

3.4 Investment period

Why choose an investment period of 10 years? Indeed, on the one hand, equity traders typically have a much shorter time horizon, sometimes even as short as a few microseconds. On the other hand, many equity investors choose a "buy-and-hold" strategy, in which shares are bought and never sold. I think a period of

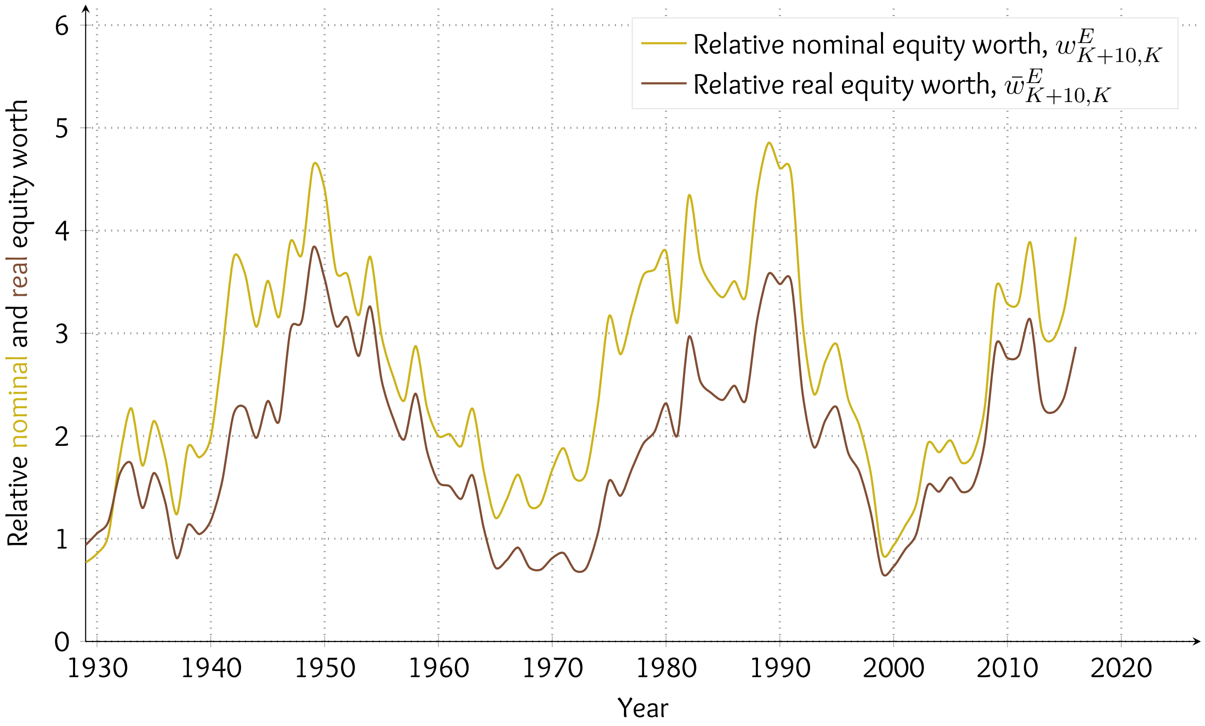

Figure 9: Histories of relative nominal equity worth, $w^E_{K+10,K}$, and relative real equity worth, $\bar{w}^E_{K+10,K}$, after investing for 10 years ($k = 10$), and *without reinvesting* dividends. The time average of the relative nominal worth is about 2.64, and the time average of the relative real worth is about 1.92. The time averaged arithmetic difference between them is about 0.72. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University.[2])

about 10 years represents some reasonable middle ground. It is sufficiently long that the influence of short-term economic fluctuations and of sentiment-driven share-trading may be neglected in this analysis. And it is sufficiently short that the analysis is able to yield meaningful results in the lifespan of a normal person.

In support of the choice of a 10 year investment period, histories for the relative real equity worth $\bar{w}^E_{K+k,K}$ were calculated for four investment periods, spanning 2 years, 5 years, 10 years, and 15 years, as shown in Figure 12 on page 19. Although the figure is somewhat cluttered, it is clear that the 2-year curve is dominated by intrinsic shorter-term influences, roughly spanning similar time periods, 2 to 3 years. Since the width of these timespans over which such influences act is about the same as that of the investment period itself, the data from which Figure 12 is derived is too coarse to account for the influences in any meaningful way. Conversely, as the investment period is increased from 2 years to 15 years, the relative real equity worth is observed to vary over timespans that increasingly exceed the width of the investment period. Indeed, both the 10-year and 15-year curves exhibit about a 30 year time interval between the two peaks centred around 1950 and 1990, three times wider than a 10 year investment period.

There is however an important sublety here. The consideration is the *choice* of timespan over which to compute the history of an investor's relative real equity worth, *not* what the driving forces might be for the variation in that history. Specifically, while the presence of the two peaks in the curves, centred around 1950 and 1990, may be interesting, extrapolating a sinusoidal variation in the curves into the future, without considering what drives the variation, is a mistake. Indeed, an investor's relative real equity worth is determined entirely by: the evolution of share prices (Figure 2 on page 10); the accumulation of dividends (Figure 4 on page 11); and inflation (Figure 1 on page 8), as described in Eqs. (14), (16) and (17). Given the somewhat erratic evolution of each of these inputs over time, it is unclear how they would conspire to ensure a continuation into the future of the sinusoidal character of the curves in Figure 12. The urge to extrapolate any trend in the investor's relative real equity worth, starting now, should therefore be resisted. All that we can reasonably conclude for now is that the effects of the driving forces, whatever they may be, span a characteristic time interval exceeding a 10 year investment period.

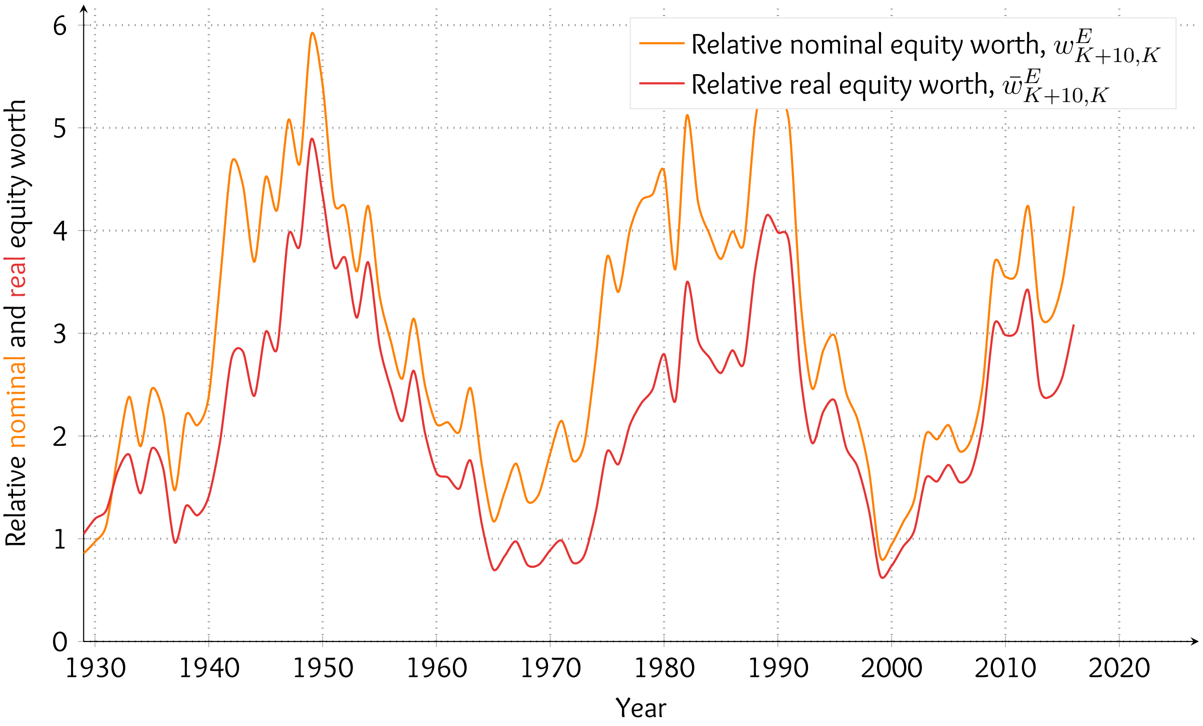

Figure 10: Histories of relative nominal equity worth, $w^E_{K+10,K}$, and relative real equity worth, $\bar{w}^E_{K+10,K}$, after investing for 10 years ($k = 10$), and *with dividends reinvested*. The time average of the relative nominal worth is about 3.01, and the time average of the relative real worth is about 2.18. The time averaged arithmetic difference between them is about 0.83. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University.[2])

In making investment decisions, investors employ a broad range of decision criteria, including: 1. a qualitative prediction of the growth sectors in an economy; 2. a requirement for a steady income stream; 3. a perception of economic (in)stability buoyed by political (in)stability; 4. short-term equity price and valuation fluctuations; 5. the relative attractiveness of different investment instruments; or simply, 6. no criteria at all, as exemplified in a buy-and-hold strategy. Some investors employ a rational and regulated methodology. Others employ a sentimental and ad hoc one. In what follows, a former methodology is developed in the arguably subjective belief that ultimately, the interplay between economic forces and investment market forces is a rational one. To develop this methology, we must broaden scope beyond that of equities, to consider an investment market much larger than the equity market, namely, the bond market.

4 Bonds

BY INVESTING IN BONDS, an investor hopes to increase redeemable monetary worth by selling the bond investment at a price higher than what it was bought for, and by obtaining payments in the form of regular and predictable coupons. Two bond markets exist, namely, the *primary bond market* and the *secondary bond market*. To encapsulate how bonds may affect an investor's redeemable monetary worth, pricing in both markets must be considered.

4.1 Primary bond market

Suppose that at some time t_K we have an amount W_K available to invest. If we choose to invest only in an interest bearing bond, then at some subsequent time t_{K+1}, the investment must be worth

$$W^B_{K+1} = B_{K+1} + C$$

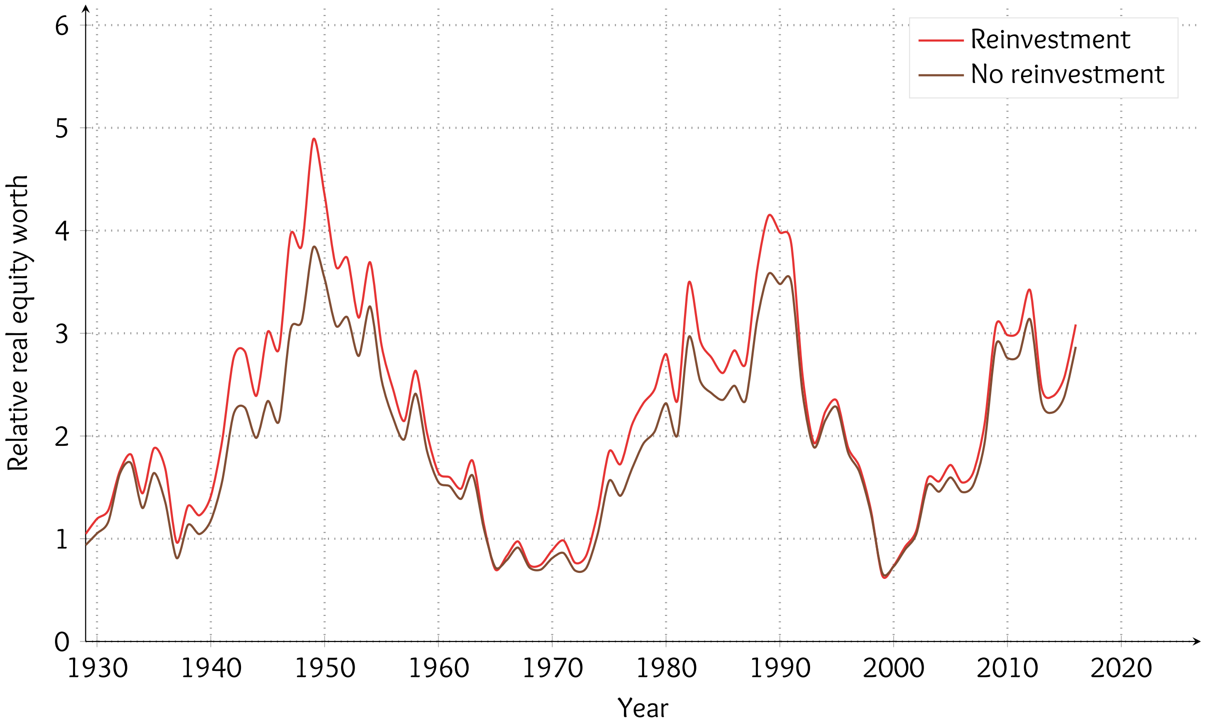

Figure 11: Histories of the relative real equity worth, $\bar{w}^E_{K+10,K}$, with reinvestment of dividends and without reinvestment of dividends, after investing for 10 years ($k = 10$). The time average of the worth with reinvestment of dividends is about 2.18, and the time average of the worth without reinvestment of dividends is about 1.92. The time averaged arithmetic difference between them is about 0.26. These curves are the real counterparts to the nominal ones in Figure 8 on page 15. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University.[2])

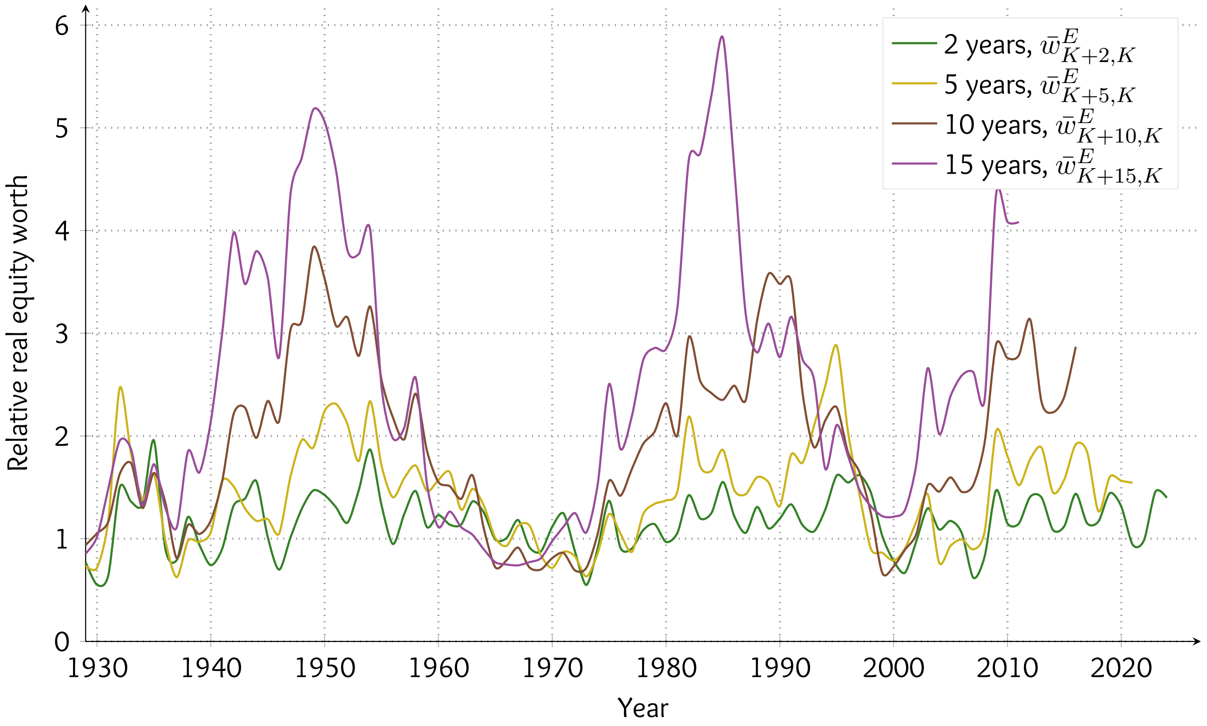

Figure 12: Histories of relative real equity worth, $\bar{w}^E_{K+k,K}$, after investing for 2 years ($k = 2$), 5 years ($k = 5$), 10 years ($k = 10$), and 15 years ($k = 15$), *without reinvesting* dividends. Based on the US S&P 500 price and dividend per share history. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University.[2])

where B_{K+1} is the value of the bond at time t_{K+1}, and C is the agreed amount paid to the bond holder by the bond issuer over the $(t_K, t_{K+1}]$ time interval.[10, 11]

But what is the value of B_{K+1}? At t_K the bond purchaser lends an amount B_K to the bond issuer. By the end of the first period, i.e., by t_{K+1}, since the purchaser has been "exposed" by an amount $B_K = W_K$, the purchaser will charge interest as compensation for the exposure. The purchaser and issuer agree that the interest time rate shall be β_K. The subscript 'K' captures the fact that the agreement was made at the time labelled t_K. And since the issuer paid the purchaser the amount C over $(t_K, t_{K+1}]$, we must have

$$B_{K+1} = \big[1 + \beta_K \Delta t_{K+1}\big] B_K - C$$

so that

$$W^B_{K+1} = \big[1 + \beta_K \Delta t_{K+1}\big] B_K - C + C$$

After two subsequent periods,

$$\begin{aligned} B_{K+2} &= \big[1 + \beta_K \Delta t_{K+2}\big] B_{K+1} - C \\ &= \big[1 + \beta_K \Delta t_{K+1}\big]\big[1 + \beta_K \Delta t_{K+2}\big] B_K - \Big[1 + \big[1 + \beta_K \Delta t_{K+2}\big]\Big] C \end{aligned}$$

After three subsequent periods,

$$\begin{aligned} B_{K+3} &= \big[1 + \beta_K \Delta t_{K+3}\big] B_{K+2} - C \\ &= \big[1 + \beta_K \Delta t_{K+1}\big]\big[1 + \beta_K \Delta t_{K+2}\big]\big[1 + \beta_K \Delta t_{K+3}\big] B_K \\ &\qquad - \Big[1 + \big[1 + \beta_K \Delta t_{K+2}\big]\big[1 + \beta_K \Delta t_{K+3}\big] + \big[1 + \beta_K \Delta t_{K+3}\big]\Big] C \end{aligned}$$

And so on for the subsequent periods. So at any arbitrary future time t_{K+k},

$$B_{K+k} = \prod_{i=1}^{k} \big[1 + \beta_K \Delta t_{K+i}\big] B_K - \Big[1 + \sum_{i=2}^{k} \prod_{j=i}^{k} \big[1 + \beta_K \Delta t_{K+j}\big]\Big] C \tag{18}$$

and

$$W^B_{K+k} = B_{K+k} + kC \tag{19}$$

4.2 Secondary bond market

Unfortunately, (19) is not the full picture. Although the bond was purchased at time t_K for an amount $B_K = W_K$, the holder may not be able to sell the bond at time t_{K+k}. The reason is that in order to sell, the holder would need to appeal to the secondary bond market—a market in which buyers are exposed to other competitive bonds. To be sure, (19) applies only if the holder does not wish to redeem his/her bond investment at t_{K+k}, but chooses instead to hold the bond until maturity at some time t_{K+N}, for some $N > k$. But doing so would limit the analysis to that of W^B_{K+N}. So to obtain an estimate of the redeemable value of an investor's bond holding, we must account for the bond's exposure to the competitive environment in the secondary bond market.

Suppose, then, that at time t_{K+k}, another bond issuer issues a bond in the primary market. The issuer offers a coupon value of D, say, and a fixed interest time rate of β_{K+k} for the duration of the bond. Again, the '$K+k$' subscript simply captures the fact that the agreement was made at the time t_{K+k}. Then following (18), at some future time t_{K+k+l} the value of this new bond is

$$A_{K+k+l} = \prod_{i=1}^{l} \big[1 + \beta_{K+k} \Delta t_{K+k+i}\big] A_{K+k} - \Big[1 + \sum_{i=2}^{l} \prod_{j=i}^{l} \big[1 + \beta_{K+k} \Delta t_{K+k+j}\big]\Big] D \tag{20}$$

If this new bond matures after, say, M periods subsequent to its issuance at the $(K+k)$-th period, then at maturity its value must be

$$A_{K+k+M} = \prod_{i=1}^{M} \big[1 + \beta_{K+k} \Delta t_{K+k+i}\big] A_{K+k} - \Big[1 + \sum_{i=2}^{M} \prod_{j=i}^{M} \big[1 + \beta_{K+k} \Delta t_{K+k+j}\big]\Big] D$$

From this, the initial issuance price of the bond at t_{K+k} in relation to its value at maturity must be

$$A_{K+k} = \frac{A_{K+k+M} + \Big[1 + \sum_{i=2}^{M} \prod_{j=i}^{M} [1 + \beta_{K+k} \Delta t_{K+k+j}]\Big] D}{\prod_{i=1}^{M} [1 + \beta_{K+k} \Delta t_{K+k+i}]} \tag{21}$$

Returning now to our first bond B. At t_{K+k}, our first bond B will mature after $(N-k)$ periods. It will (hopefully) still provide $(N-k)$ coupon payments each valued at C. And at maturity, it will redeem an amount B_{K+N} to the original purchaser. So bond A will be a natural competitor to bond B if A has the same attributes. That is, to capture the competition between bonds A and B we must set

$$K + k + M = K + N \implies M = N - k$$

$$D = C$$

$$A_{K+k+M} = B_{K+N}$$

Therefore, in order for bonds A and B to compete, the initial issuance price of bond A_{K+k} at t_{K+k} must be, using (21),

$$A_{K+k} = \frac{B_{K+N} + \Big[1 + \sum_{i=2}^{N-k} \prod_{j=i}^{N-k} [1 + \beta_{K+k} \Delta t_{K+k+j}]\Big] C}{\prod_{i=1}^{N-k} [1 + \beta_{K+k} \Delta t_{K+k+i}]} \tag{22}$$

The essential observation here is that since bonds A and B are now similarly attributed at t_{K+k}, they compete against each other. That means that the sellable value of bond B at t_{K+k} must equal the issuance price of bond A at t_{K+k}, namely, that in (22). And so the investment value at t_{K+k} for the purchaser of the original bond B must be

$$W^B_{K+k} = A_{K+k} + kC \tag{23}$$

that is obstensibly different from (19).

It is noteworthy that using (20) to calculate the value for A_{K+N} by setting $l = N - k$ in (20), and then using (22) for the issuance price A_{K+k}, gives $A_{K+N} = B_{K+N}$, as expected. This means from (23) that

$$W^B_{K+N} = B_{K+N} + NC$$

and that agrees with (19) when we set $k = N$ in (19).

4.3 Coupon bonds

Thus far, we have not considered how the bond issuer and the bond purchaser agree on fixing the initial issuance B_K, the coupon value C, the bond term N, the interest time rate β_K, and the value of the bond at maturity B_{K+N}. Different types of bonds fix these various quantities differently.[10, 11] Here we now restrict our analysis to that of coupon bonds.

A coupon bond is one in which the bond issuer agrees to reimburse the bond purchaser the initial borrowed amount at maturity. So setting

$$k = N \text{ and } B_{K+N} = B_K \tag{24}$$

in (18) gives

$$B_K = \prod_{i=1}^{N} [1 + \beta_K \Delta t_{K+i}] B_K - \Big[1 + \sum_{i=2}^{N} \prod_{j=i}^{N} [1 + \beta_K \Delta t_{K+j}]\Big] C$$

So if the coupon bond issuer wishes to borrow an amount B_K at t_K, and if the issuer and the purchaser agree that the interest time rate shall be fixed at β_K for the N periods spanning $(t_K, t_{K+N}]$, then the coupon must be

$$C = \frac{\displaystyle\prod_{i=1}^{N} [1 + \beta_K \Delta t_{K+i}] - 1}{1 + \displaystyle\sum_{i=2}^{N} \prod_{j=i}^{N} [1 + \beta_K \Delta t_{K+j}]} \cdot B_K \tag{25}$$

Substituting (24) and (25) into (22) gives, in all its gory detail,

$$A_{K+k} = \frac{1}{\displaystyle\prod_{i=1}^{N-k} b_{K+k,i}} \left[1 + \frac{\left(1 + \displaystyle\sum_{i=2}^{N-k} \prod_{j=i}^{N-k} b_{K+k,j} \right) \left(\displaystyle\prod_{l=1}^{N} b_{K,l} - 1 \right)}{1 + \displaystyle\sum_{i=2}^{N} \prod_{j=i}^{N} b_{K,j}} \right] B_K \tag{26}$$

with

$$b_{u,v} \equiv 1 + \beta_u \Delta t_{u+v} \qquad \text{for any } u, v$$

Calculating the redeemable nominal value at time t_{K+k} of a coupon bond investment using (23) with (26) requires to track the set $\{\Delta t_{K+i} \mid i = 1, 2, \ldots N\}$ of time intervals during which the bond coupons (25) are paid by the bond issuer. The analysis leading to (25) and (26) has not assumed these time intervals to be of equal size. However, in reality they most probably are, typically one month or one year. Therefore, substituting the condition

$$\Delta t_{u+v} = \Delta t \qquad \text{for all } u, v$$

in (25) and (26) gives

$$\begin{aligned} C &= \frac{(1 + \beta_K \Delta t)^N - 1}{1 + \displaystyle\sum_{i=2}^{N} (1 + \beta_K \Delta t)^{N-i+1}} \cdot B_K \\ &= \frac{(1 + \beta_K \Delta t)^N - 1}{1 + (1 + \beta_K \Delta t) \displaystyle\sum_{i=0}^{N-2} (1 + \beta_K \Delta t)^i} \cdot B_K \\ &= \frac{(1 + \beta_K \Delta t)^N - 1}{1 + (1 + \beta_K \Delta t) \left(\dfrac{1 - (1 + \beta_K \Delta t)^{N-1}}{1 - (1 + \beta_K \Delta t)} \right)} \cdot B_K \\ &= \beta_K \Delta t B_K \end{aligned} \tag{27}$$

And after some similar algebraic manipulation,

$$A_{K+k} = \left[\frac{\beta_K}{\beta_{K+k}} + \left(1 - \frac{\beta_K}{\beta_{K+k}} \right) \frac{1}{(1 + \beta_{K+k} \Delta t)^{N-k}} \right] B_K \tag{28}$$

The tendency for bond prices in the secondary bond market to vary reciprocally with variation in the prevailing bond rate is quantitatively revealed in (28). For as β_{K+k} increases, A_{K+k} decreases. A long-term history of the US 10–year treasury constant maturity interest rate, β_K, is shown in Figure 13.

Having now established that the initial bond B investor is able to sell his/her coupon bond at time t_{K+k} in the competitive secondary bond market at a price A_{K+k}, the investor's redeemable investment at t_{K+k} must be worth, using (23), (27) and (28), and the fact that $W_K = B_K$,

$$W^B_{K+k} = \left[k \beta_K \Delta t + \frac{\beta_K}{\beta_{K+k}} + \left(1 - \frac{\beta_K}{\beta_{K+k}} \right) \frac{1}{(1 + \beta_{K+k} \Delta t)^{N-k}} \right] W_K \tag{29}$$

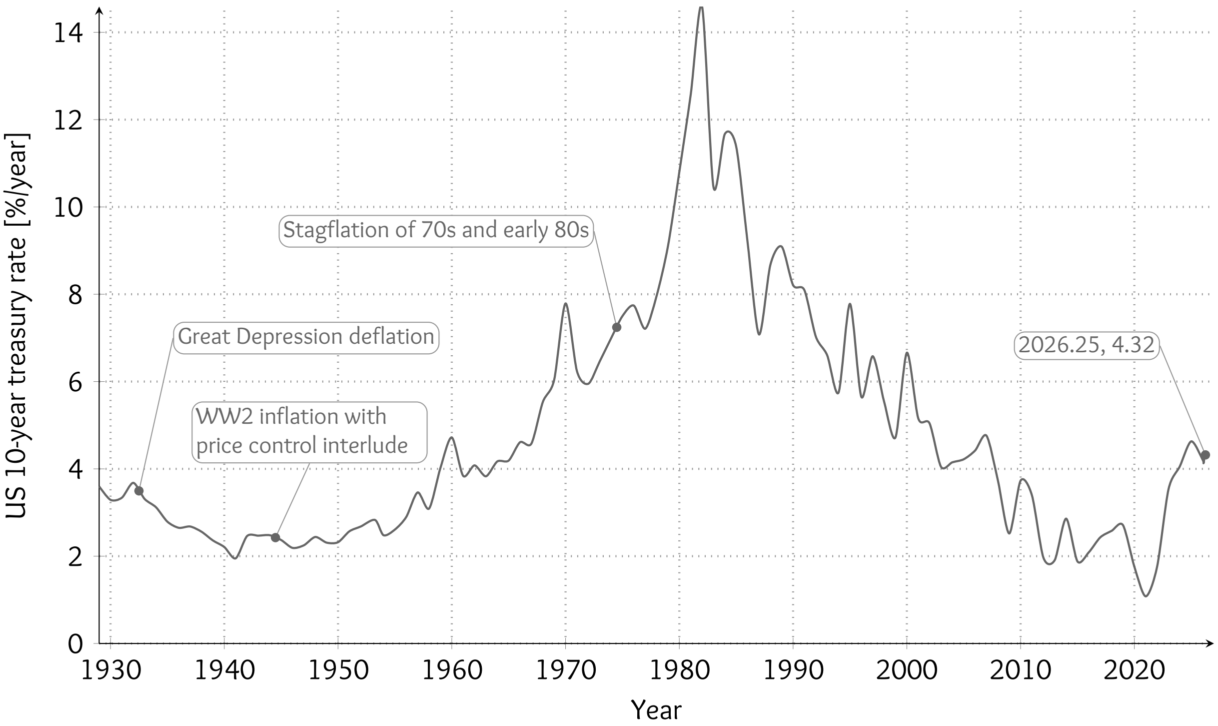

Figure 13: History of the US 10–year treasury constant maturity interest rate, β_K. The long-term time averaged rate is 4.77% p.a. (Raw data sources: Board of Governors of the Federal Reserve System,[12] Yale University.[2])

And the bond investor's relative nominal worth at t_{K+k} is

$$\boxed{w^B_{K+k,K} \equiv \frac{W^B_{K+k}}{W_K} = k\beta_K\Delta t + \frac{\beta_K}{\beta_{K+k}} + \left(1 - \frac{\beta_K}{\beta_{K+k}}\right)\frac{1}{(1+\beta_{K+k}\Delta t)^{N-k}}} \tag{30}$$

It is worth nothing that at the time of bond maturity t_{K+N}, (30) reduces to $w^B_{K+N,K} = 1 + N\beta_K\Delta t$, as expected.

Unfortunately, once again, as with equity investments (Subsection 3.3 on page 15), inflation erodes the purchasing ability attributable to W^B_{K+k} at t_{K+k}. To account for the erosion, we must focus on an investment worth at t_{K+k}, based on pricing at t_J for some $J > 0$. That is, following the analysis in Subsection 3.3 on page 15, we must calculate the *real bond investment worth* at t_{K+k}, using (9) on page 7, as

$$\bar{W}^B_{K+k,J} = \bar{W}^B_{J+(K-J+k),J} = \frac{W^B_{J+(K-J+k)}}{\prod\limits_{l=1}^{K-J+k} (1+\alpha_{J+l}\Delta t_{J+l})}$$

Following (17), the bond investor's *relative real bond worth* at t_{K+k} relative to that at t_K and based on pricing at t_J is then

$$\boxed{\begin{aligned}\bar{w}^B_{K+k,J} \equiv \frac{\bar{W}^B_{K+k,J}}{\bar{W}^B_{K,J}} &= \frac{1}{\prod\limits_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})} \cdot \frac{W^B_{K+k}}{W^B_K} \\ &= \frac{w^B_{K+k,K}}{\prod\limits_{l=1}^{k}(1+\alpha_{K+l}\Delta t_{K+l})} \\ &= \bar{w}^B_{K+k,K}\end{aligned}} \tag{31}$$

This shows that once again, along with (17) for the relative equity real worth, the relative real bond worth at some time t_{K+k} relative to that at t_K is sensitive to the eroding effects of inflation only over the $(t_K, t_{K+k}]$ time interval. Therefore, in what follows, we may work with $\bar{w}^B_{K+k,K}$ rather than with $\bar{w}^B_{K+k,J}$.

4.4 Secondary bond market only

Unfortunately, (29), (30) and (31) are still not the full picture. The nominal bond worth W^B_{K+k} in (29) was obtained at t_{K+k} by purchasing a bond at t_K in the primary market, possibly with a view to selling the bond before term in the secondary market. While these transactions obviously do happen from time to time, they are less frequent than ones in which both purchasing and selling is done in the secondary market. Let us then focus on the secondary bond market.

Of course, there is no single strategy for investing in the secondary bond market. One strategy could be to purchase a coupon bond, and then to hold it until the bond reaches maturity. That is, until $k = N$. This strategy effectively constrains the investment period's timespan to equal the bond's time lapse till maturity. But it is evident from (28) that at maturity, i.e., at $t_{K+k} = t_{K+N}$, the bond's value must be $A_{K+N} = B_K$, and from (30), the investor's relative nominal bond worth must be $w^B_{K+N,K} = 1 + N\beta_K\Delta t$. So under such an investment strategy, the investment becomes immune to variations in prevailing bond interest rates over time.

An alternative simulation strategy is considered here. It is one that retains a sensitivity to prevailing interest rate variations, and that captures an investor's possible engagement with an investment fund, such as an exchange-traded fund (ETF). ETFs transact in the secondary bond market on behalf of investors. To simulate transactions carried out by the ETF in the secondary bond market, we consider here a strategy in which the bond's time lapse till maturity is rendered constant over any investment period. We develop the model incrementally, beginning at the current time t_K.

Not reinvesting coupons. At time t_K, suppose that we as a bond investor have an amount W_K to invest in an interest bearing coupon bond. The ETF buys a coupon bond B on our behalf. Being a coupon bond that matures after N periods, say, we have from (24) that $W_K = W^B_K = B_K = B_{K+N}$. And from (27), the investor's coupons will be valued at

$$C_K = \beta_K \Delta t B_K$$

with β_K being the prevailing interest time rate in the secondary market at t_K. Here it is important to note that the 'K' subscript attached to C_K simply denotes that the coupon was fixed at t_K, applicable at each period in the entire $(t_K, t_{K+N}]$ interval. By the beginning of the subsequent time, t_{K+1}, the investor would have received that first C_K coupon.

At time t_{K+1}, from (28), a reasonable value for the bond in the secondary market must be

$$B_{K+1} = \gamma_{K+1} B_K \tag{32}$$

where

$$\gamma_u \equiv \frac{\beta_{u-1}}{\beta_u} + \left(1 - \frac{\beta_{u-1}}{\beta_u}\right)\frac{1}{(1+\beta_u\Delta t)^{N-1}} \qquad \text{for any } u > 0 \tag{33}$$

Note that γ_u here is just the large bracketed coefficient in (28) with the exponent $(N-k)$ set to $(N-1)$. From (30), our corresponding relative nominal bond worth at t_{K+1} relative to that at t_K must be

$$w^B_{K+1,K} = \frac{C_K + B_{K+1}}{B_K} = \beta_K\Delta t + \gamma_{K+1}$$

The ETF then sells the bond that was bought on behalf of the investor at t_K, and uses the proceeds to purchase a new coupon bond that has the same time lapse till maturity. That is, the new bond has an initial value B_{K+1} at t_{K+1}. But it will now mature at t_{K+1+N}, and it is now contracted to reward the investor N constant coupons each valued at

$$C_{K+1} = \beta_{K+1}\Delta t B_{K+1}$$

at each period in the $(t_{K+1}, t_{K+1+N}]$ interval. By the beginning of the subsequent time, t_{K+2}, the investor would have received that second coupon, C_{K+1}.

At time t_{K+2}, from (28), a reasonable value for the bond in the secondary market at t_{K+2} must be

$$B_{K+2} = \gamma_{K+2} B_{K+1} = \gamma_{K+2}\gamma_{K+1} B_K \tag{34}$$

From (30), our corresponding relative nominal bond worth at t_{K+2} relative to that at t_K must now be

$$\begin{aligned} w^B_{K+2,K} &= \frac{C_K + C_{K+1} + B_{K+2}}{B_K} \\ &= \beta_K \Delta t + \frac{\beta_{K+1}\Delta t B_{K+1} + \gamma_{K+2} B_{K+1}}{B_K} \\ &= \beta_K \Delta t + (\beta_{K+1}\Delta t + \gamma_{K+2})\gamma_{K+1} \end{aligned}$$

The ETF then sells the bond that was bought at t_{K+1}, and uses the proceeds to purchase a new coupon bond that, again, has the same time lapse till maturity. That is, the new bond has an initial value B_{K+2} at t_{K+2}. It will mature at t_{K+2+N}, and it is contracted to reward the investor N constant coupons each valued at

$$C_{K+2} = \beta_{K+2}\Delta t B_{K+2}$$

at each period in the $(t_{K+2}, t_{K+2+N}]$ interval.

By continuing in this manner for subsequent times t_{K+3}, t_{K+4},…, it is easy to deduce that a reasonable value for the bond at t_{K+k} must be

$$B_{K+k} = \gamma_{K+k} B_{K+k-1} = \prod_{j=1}^{k} \gamma_{K+j} B_K \qquad \text{for } k > 0 \tag{35}$$

And the investor's corresponding relative nominal bond worth at t_{K+k} must be

$$\boxed{w^B_{K+k,K} = \sum_{i=0}^{k-1} \beta_{K+i}\Delta t \prod_{j=1}^{i} \gamma_{K+j} + \prod_{j=1}^{k} \gamma_{K+j} \qquad \text{for } k > 0}$$

Equation (35) includes a recursive form for B_{K+k}. A corresponding recursive form for $w^B_{K+k,K}$, is obtained as

$$\begin{aligned} w^B_{K+k,K} &= \sum_{i=0}^{k-2} \beta_{K+i}\Delta t \prod_{j=1}^{i} \gamma_{K+j} + \beta_{K+k-1}\Delta t \prod_{j=1}^{k-1} \gamma_{K+j} + \prod_{j=1}^{k} \gamma_{K+j} \\ &= \sum_{i=0}^{k-2} \beta_{K+i}\Delta t \prod_{j=1}^{i} \gamma_{K+j} + \prod_{j=1}^{k-1} \gamma_{K+j} - \prod_{j=1}^{k-1} \gamma_{K+j} + \beta_{K+k-1}\Delta t \prod_{j=1}^{k-1} \gamma_{K+j} + \prod_{j=1}^{k} \gamma_{K+j} \\ &= \Big(\sum_{i=0}^{k-2} \beta_{K+i}\Delta t \prod_{j=1}^{i} \gamma_{K+j} + \prod_{j=1}^{k-1} \gamma_{K+j}\Big) + \Big(\beta_{K+k-1}\Delta t + \gamma_{K+k} - 1\Big) \prod_{j=1}^{k-1} \gamma_{K+j} \\ &= w^B_{K+k-1,K} + \Big(\beta_{K+k-1}\Delta t + \gamma_{K+k} - 1\Big)\pi_{K,k-1} \end{aligned}$$

where

$$\pi_{u,v} = \prod_{j=1}^{v} \gamma_{u+j} = \gamma_{u+v}\pi_{u,v-1} \qquad \text{for any } u, v$$

and to repeat (33),

$$\gamma_{K+l} \equiv \frac{\beta_{K+l-1}}{\beta_{K+l}} + \left(1 - \frac{\beta_{K+l-1}}{\beta_{K+l}}\right) \frac{1}{(1+\beta_{K+l}\Delta t)^{N-1}} \qquad \text{for any } l > 0$$

A recursive algorithm to compute $w^B_{K+k,K}$ is therefore

```
$w^B_{K+0,K} \equiv 1$
$\pi_{K,0} \equiv 1$
for $i = 1, 2, \ldots, k$
    $w^B_{K+i,K} = w^B_{K+i-1,K} + \big(\beta_{K+i-1}\Delta t + \gamma_{K+i} - 1\big)\pi_{K,i-1}$
    $\pi_{K,i} \equiv \gamma_{K+i}\pi_{K,i-1}$
```

(36)

Reinvesting coupons. Instead of simply receiving the coupon payments through the purchase of a bond, the investor may choose to use the proceeds to increase his/her bond investment. This is similar to an equity investor using the proceeds from dividend distributions to purchase additional equities. As before, we develop the model incrementally, beginning at the current time t_K.

At time t_K, suppose that we as a bond investor have an amount W_K to invest in an interest bearing coupon bond. The ETF buys a coupon bond B on our behalf. Then at t_K

$$W_K = W^B_K = B_K$$

By the beginning of the subsequent time period at t_{K+1}, the bond investor would have received a coupon payment

$$C_K = \beta_K \Delta t B_K = \beta_K \Delta t W^B_K$$

At time t_{K+1}, if we wish to sell the bond at t_{K+1}, we would have to appeal to the secondary bond market in which other bonds are being bought and sold. So following (32), the bond would be sellable at a price

$$B_{K+1} = \gamma_{K+1} B_K = \gamma_{K+1} W^B_K$$

with γ_{K+1} given by (33). Our bond is then sold by the ETF, and the proceeds are used to purchase a new coupon bond that has the same time lapse till maturity. Except, in contrast with (34), we choose to use the coupon just received to help purchase the new bond. Immediately after the purchase, the investor's redeemable nominal bond worth at time t_{K+1} must therefore be

$$W^B_{K+1} = B_{K+1} + C_K$$

Note that the value of the new bond just purchased is not B_{K+1}, but rather W^B_{K+1}. This is because the coupon C_K just received contributes to the purchase.

At time t_{K+2}, the ETF once again employs on our behalf the same sell-then-repurchase tactic. The ETF sells the bond at a price

$$B_{K+2} = \gamma_{K+2} W^B_{K+1}$$

in the secondary market immediately after receiving the coupon

$$C_{K+1} = \beta_{K+1} \Delta t W^B_{K+1}$$

The ETF then purchases a new bond in the secondary market using the proceeds of the sale and the coupon just received. So the investor's redeemable nominal bond worth at time t_{K+2} must be

$$\begin{aligned}
W^B_{K+2} &= B_{K+2} + C_{K+1} \\
&= \gamma_{K+2} W^B_{K+1} + \beta_{K+1}\Delta t W^B_{K+1} \\
&= (\beta_{K+1}\Delta t + \gamma_{K+2}) W^B_{K+1} \\
&= (\beta_{K+1}\Delta t + \gamma_{K+2})(C_K + B_{K+1}) \\
&= (\beta_{K+1}\Delta t + \gamma_{K+2})(\beta_K \Delta t + \gamma_{K+1}) W^B_K
\end{aligned}$$

By continuing in this manner for subsequent times t_{K+3}, $t_{K+4}\ldots$, it is easy to deduce that the investor's redeemable nominal bond worth at t_{K+k} must be

$$\begin{aligned} W^B_{K+k} &= (\beta_{K+k-1}\Delta t + \gamma_{K+k}) W^B_{K+k-1} \\ &= \prod_{i=1}^{k} (\beta_{K+i-1}\Delta t + \gamma_{K+i}) W^B_K \qquad \text{for any } k > 0 \end{aligned}$$

And so the investor's redeemable relative nominal bond worth at t_{K+k} must be

$$\boxed{\begin{aligned} w^B_{K+k,K} = \frac{W^B_{K+k}}{W^B_K} &= (\beta_{K+k-1}\Delta t + \gamma_{K+k}) w^B_{K+k-1,K} \\ &= \prod_{i=1}^{k} (\beta_{K+i-1}\Delta t + \gamma_{K+i}) \qquad \text{for any } k > 0 \end{aligned}} \tag{37}$$

with γ_{K+i} given by (33). Equation (37) is the bond counterpart to (16) on page 14 for the reinvestment of dividends in an equity investment.

Long-term histories of the corresponding relative real bond worth $\bar{w}^B_{K+10,K}$ as per (31) without reinvestment of coupon receipts (Algorithm (36)) and with reinvestment of coupon receipts (Equation (37)) are shown in Figure 14. And as with the results shown in Figure 11 on page 19, the investment period is set to 10 years ($k = 10$). Note that the vertical extent of the y-axis has intentionally been set to match that in Figures 8, 9, 10, 11 and 12.

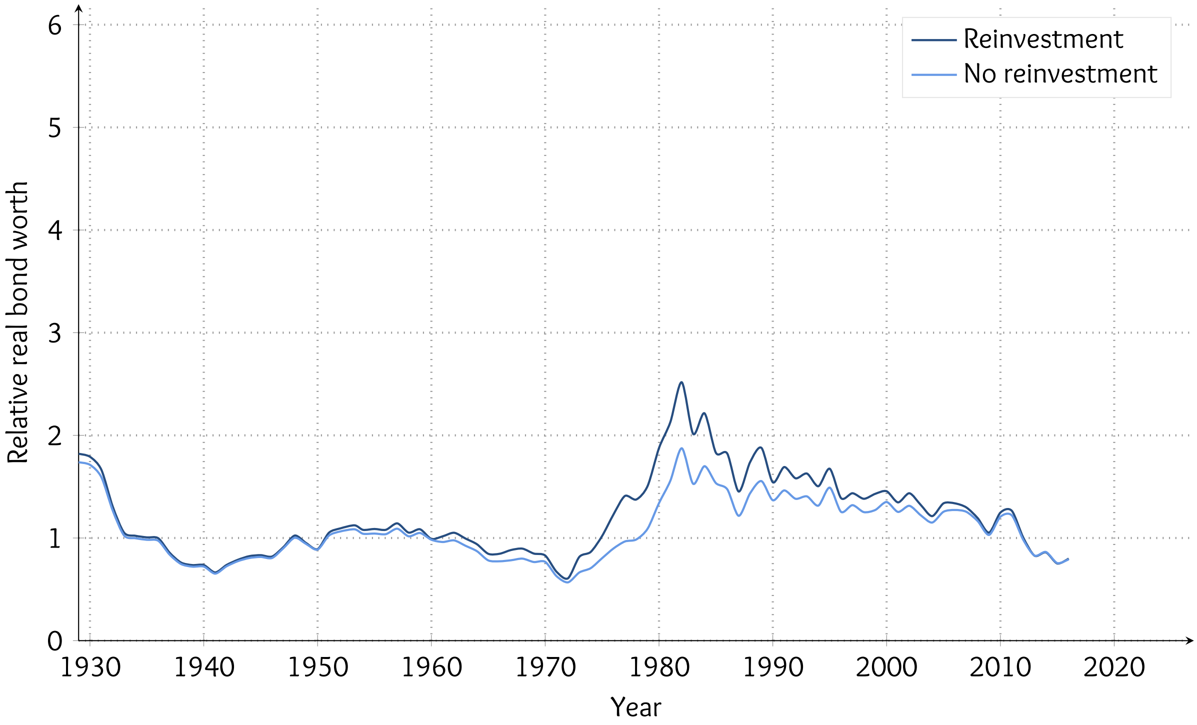

Figure 14: History of the relative real bond worth, $\bar{w}^B_{K+10,K}$, with reinvestment of coupon receipts and without reinvestment of coupon receipts after investing for 10 years ($k = 10$). The time average $\langle \bar{w}^B \rangle$ of the worth with reinvestment of coupon receipts is about 1.2, and the time average of the worth without reinvestment of coupon receipts is about 1.08. The time averaged arithmetic difference between them is about 0.12. (Raw data sources: Board of Governors of the Federal Reserve System,[12] Yale University.[2])

5 Equities versus bonds

HOW ARE WE TO MEASURE the relative attractiveness of equities and bonds? The notion of *relative real worth* is useful here. It is reasonable to assert that the instrument yielding a higher relative real worth over some specified investment period is deemed more attractive at the start of such a period.

Suppose, once again, that the time at the start of such a period is t_K, and that the investment period is the time spanned by the $(t_K, t_{K+k}]$ interval, for some $k > 0$. If the indices K and k are such that the time interval resides fully in the past, then the impact of decisions taken at t_K for the ensuing interval may be analysed exactly. At t_K, an investment in equities instead of in bonds over $(t_K, t_{K+k}]$ would have been preferable if the investor's *change in relative real equity worth*, $\Delta\bar{w}^E_{K+k,K}$, by investing in equities was to have exceeded the investor's *change in relative real bond worth*, $\Delta\bar{w}^B_{K+k,K}$, by investing in bonds over $(t_K, t_{K+k}]$. That is, when facing an investment decision at the historical time t_K, investing in equities would have been preferable to investing in bonds if by some later historical time t_{K+k},

$$\Delta\bar{w}^E_{K+k,K} \equiv \frac{\bar{W}^E_{K+k,K} - \bar{W}^E_{K,K}}{\bar{W}^E_{K,K}} = \bar{w}^E_{K+k,K} - 1 \;>\; \bar{w}^B_{K+k,K} - 1 = \frac{\bar{W}^B_{K+k,-}\bar{W}^B_{K,K}}{\bar{W}^B_{K,K}} \equiv \Delta\bar{w}^B_{K+k,K}$$

for some selected k. Therefore, equities would have been favoured over bonds at t_K if

$$\boxed{\bar{w}^E_{K+k,K} \;-\; \bar{w}^B_{K+k,K} > 0 \qquad \text{for some selected} \;\; k > 0}$$

A long-term history of this difference between the relative real worths is shown in Figure 15. As before, the investment period is set to 10 years ($k = 10$). In calculating the two relative real worths, dividends and coupons were both assumed to have been reinvested. What does this long-term data spanning nearly a century reveal? Firstly, for an investment period spanning 10 years, investing in bonds would have been preferable over equities for only 13.64 percent of the time. And secondly, the long-term time averaged arithmetic difference between the relative real equity worth and the relative real bond worth is about 0.97. I think both these findings are significant.

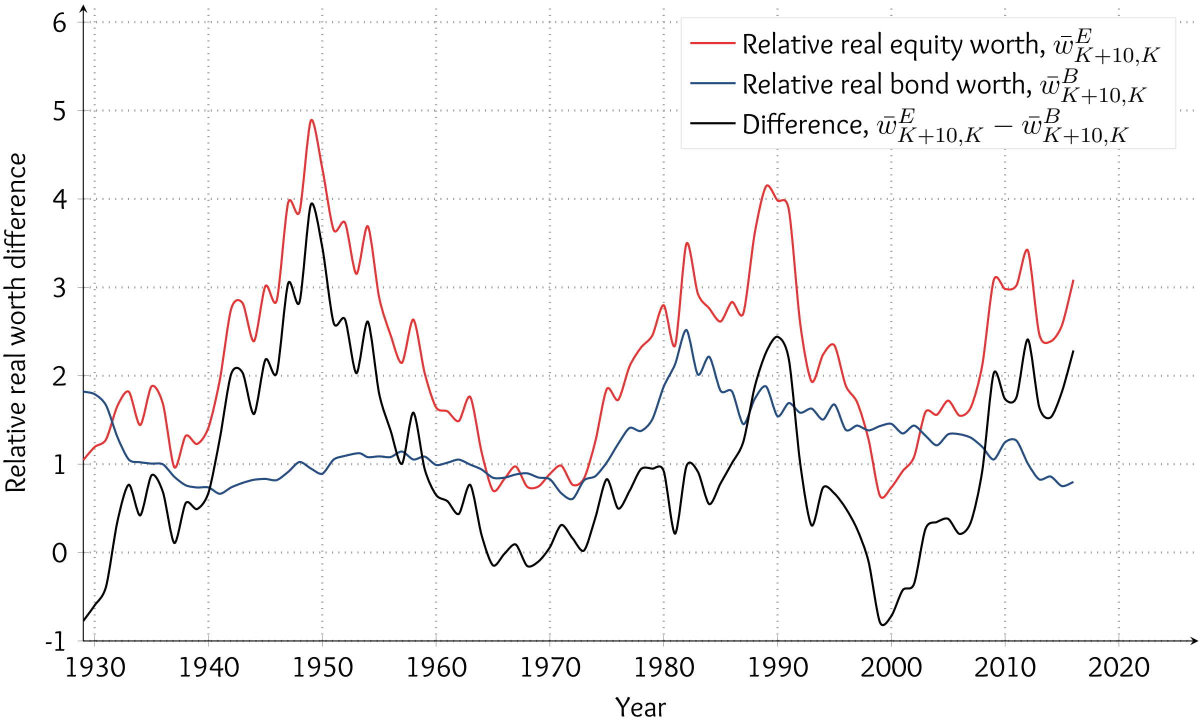

Figure 15: History of the difference between the relative real equity worth $\bar{w}^E_{K+10,K}$ (*with dividends reinvested*) and the relative real bond worth $\bar{w}^B_{K+10,K}$ (*with coupons reinvested*) after investing for 10 years ($k = 10$). The time average of the relative real equity worth is about 2.18, and the time average of the relative real bond worth is about 1.2. The time averaged arithmetic difference between them is about 0.97. An investment in bonds would have been preferable over equities for only 13.64 percent of the time, that is, when the difference curve is below 0. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Board of Governors of the Federal Reserve System,[12] Yale University.[2])

6 Predictive model

THE PREVIOUS SECTION clearly demonstrates a long-term preference for equities over bonds. So we turn our attention now to equity investing. What we seek, ultimately, is a sensible signalling strategy to guide us when best to buy, hold or sell equities. In this section, we explore four such strategies. The first strategy is the conventional "buy-and-hold" one. No matter how cheap or expensive equities may seem at any present time, they are never sold. A second strategy posits that if equities *presently* seem inexpensive relative to some long-term measure of expensiveness, then equities are deemed attractive, presently. This strategy is widely used by analysts. A third strategy considers the *future* expensiveness of equities relative to a long-term measure of expensiveness. Finally, a fourth strategy combines both the second and third one in a qualitative way. By applying these four strategies to the historical data, it will be shown here that the third strategy is most effective.

6.1 Strategy 1—No signalling, so "Buy and Hold"

Equities are bought and never sold, regardless of prevailing market conditions. This strategy is often motivated by the belief that an inability to predict the future means that no judgements can be made about the expensiveness of present-day equities. In this case, the signalling ruleset reduces to simply hold equities.

Algorithm 1 Strategy 1—No signalling, so "Buy and Hold"

```
For any time $t_K$:
    Hold equity
```

6.2 Strategy 2—Signalling based only on present-day equity expensiveness

In this strategy, present-day equities are attractive when they are deemed inexpensive relative to a measure of their long-term expensiveness. Specifically, equities at the present time t_K are attractive when their present PE ratio is less than the long-term time averaged value. That is, when

$$\rho_K < \langle \rho \rangle \qquad \text{for any } K > 0$$

The following simple ruleset implements this strategy to signal when to buy, hold or sell equities at the present time t_K.

Algorithm 2 Strategy 2—Signalling based only on present-day equity expensiveness

```
For any time $t_K$:
    $\langle \rho \rangle = 17.5$ years
    $\lambda = 0.1$
    if $\rho_K < (1 - \lambda) \langle \rho \rangle$ then
        Buy equity
    else if $\rho_K \leqslant (1 + \lambda) \langle \rho \rangle$ then
        Hold equity
    else
        Sell equity
```

6.3 Strategy 3—Signalling based on future equity expensiveness

In this third strategy, present-day equities are attractive when it is anticipated that they will in future be deemed inexpensive relative to a measure of long-term expensiveness. This strategy is ostensibly forward-looking. To quantify future expensiveness, we require a model of the future that is based on sensible extrapolations from the past and present. And to obtain such extrapolations, we shall appeal to the notion of the relative real equity worth and to historical data.

6.3.1 Relative real equity worth

We begin with the relative real equity worth. Equities are to be deemed attractive when it is anticipated that they will offer the investor a relative real equity worth in the future that exceeds a relative real worth from bonds. That is, equities are to be deemed attractive at some present time t_K when it is anticipated that the relative real equity worth over the forthcoming $(t_K, t_{K+k}]$ time interval will satisfy

$$\bar{w}^E(t_{K+k}; t_K) \;\geqslant\; \bar{w}_{\text{min}} \;>\; \bar{w}^B(t_{K+k}; t_K) \qquad \text{for any } K, k > 0 \text{ and for some } \bar{w}_{\text{min}}$$

Equations (16), (17), (31) and (37) refer. This obviously requires estimating a future relative real bond worth, $\bar{w}^B(t_{K+k}; t_K)$. Doing so is indeed a worthwhile exercise. But for now, we work instead with its long-term time average, $\langle \bar{w}^B \rangle$, thereby giving the condition for equities being more attractive than bonds as

$$\boxed{\bar{w}^E(t_{K+k}; t_K) \;\geqslant\; \bar{w}_{\text{min}} \;>\; \langle \bar{w}^B \rangle \qquad \text{for any } K, k > 0 \text{ and for some } \bar{w}_{\text{min}}} \tag{38}$$

Note that here we write '$\bar{w}^E(t_{K+k}; t_K)$' instead of '$\bar{w}^E_{K+k,K}$' to denote that we are dealing with unknown future quantities that we will need to functionally estimate. Using (17) with (16) gives

$$\bar{w}^E(t_{K+k}; t_K) \;=\; \frac{\displaystyle\prod_{i=1}^{k} \left(1 + \frac{d(t_{K+i})\Delta t}{e(t_{K+i})\rho(t_{K+i})}\right) \frac{e(t_{K+k})\rho(t_{K+k})}{e_K \rho_K}}{\displaystyle\prod_{l=1}^{k} (1 + \alpha(t_{K+l})\Delta t)} \;\geqslant\; \bar{w}_{\text{min}} \;>\; \langle \bar{w}^B \rangle$$
$$\text{for any } K, k > 0 \tag{39}$$

where we have identified $E(t) = e(t)\rho(t)$, $e(t_K) = e_K$ and $\rho(t_K) = \rho_K$ exactly. Note too that we have assumed $\Delta t_l = \Delta t$ for all l.

Stationed at t_K now, if we happen to know future values for $d(t)$, $e(t)$, $\alpha(t)$ and $\rho(t)$ for $t \in (t_K, t_{K+k}]$, then we would of course be able to use (39) to predict $\bar{w}^E(t_{K+k}; t_K)$. But since we do not know such future values, what can we do? Scrutiny of the historical data reveals some useful insights that will help us to proceed.

Insight 1 Figure 1 on page 8 shows the US inflation rate fluctuating roughly between 0% and 5% since about 1985. And the characteristic time period for those fluctuations is less than 10 years. It is reasonable to assume that this functuation of the US inflation rate will be sustained into the foreseeable future.

Insight 2 Figure 3 on page 11 shows the nominal earning rate per share exhibiting roughly an exponential evolution over most of the last century. Again, the characteristic time period of deviation from exponential behaviour is less than 10 years.

Insight 3 Figure 6 on page 12 shows a steady decline of the dividend payout ratio, $r_K \equiv d_K/e_K$, of the S&P 500 index over the span of about the last century.

Insight 4 Figure 7 on page 13 shows the price per unit of earning rate ratio, or simply, the PE ratio $\rho_K = E_K/e_K$, exhibiting little discernable behaviour over about the last century.

Insight 5 Insight 4 notwithstanding, Figure 7 and Figure 10 on page 18 hint at a loose relationship between the S&P 500 index's PE ratio and a relative real equity worth. In particular, relative real equity worth over a period tends to be high whenever the PE ratio rises over that same period. And conversely.

Insight 6 The dividend payout rate per share d, the earning rate per share e, and the inflation rate α, are all variables that should obstensibly not depend on an equity's price per share. Admittedly, however, this insight is an assumption that has not been inferred from the historical data.

6.3.2 Estimating the earning rate per share

With Insight 2 in mind, to fit an exponential function to the historical data for the earning rate per share, we insist that the function passes through three carefully selected data points (t_1, e_1), (t_2, e_2) and (t_3, e_3). Starting with the general exponential form

$$e(t) = A + B\mathrm{e}^{Ct} \tag{40}$$

for some constants A, B and C, it is straightforward to obtain

$$e(t) = \frac{\mathrm{e}^{Ct} - \mathrm{e}^{Ct_1}}{\mathrm{e}^{Ct_2} - \mathrm{e}^{Ct_1}}(e_2 - e_1) + e_1$$

Unfortunately, it is not well-posed for numerical computation because the exponents of the exponentials can be large. An alternative form, therefore, is

$$e(t; s) = \frac{\mathrm{e}^{s(t-t_1)/(t_2-t_1)} - 1}{\mathrm{e}^s - 1}(e_2 - e_1) + e_1 \tag{41}$$

for some parameter $s > 0$. Note that $e(t; s)$ in this form still satisfies $e(t_1; s) = e_1$ and $e(t_2; s) = e_2$ for any s. How do we now fix the "intensity factor" s to ensure that $e(t_3; s) = e_3$? Define the function

$$f(s) = e(t_3; s) - e_3 \tag{42}$$

Then the root of $f(s) = 0$ will be the value of s for which $e(t_3; s) = e_3$. The well known and numerically stable Bisection Method will be used to find an approximation to the root of

$$f(s) = e(t_3; s) - e_3 = 0 \tag{43}$$

6.3.3 Estimating the dividend payout ratio

With Insight 3 in mind and following the earning rate per share, we will fit an exponential decay function to the historical dividend payout ratio data. But why not simply fit a linear function, which is after all simpler? The reason is that it is reasonable to assume that the dividend payout ratio cannot be negative. Dividend payment rate per share must always be a non-negative fraction of the earning rate per share.

Let $r(t)$ be the estimate of the dividend payout ratio at some time t. Analogous with (41) and (43), we insist that $r(t)$ passes through three carefully selected data points (t'_1, r_1), (t'_2, r_2) and (t'_3, r_3). Then the function

$$r(t; u) = \frac{\mathrm{e}^{u(t-t'_1)/(t'_2-t'_1)} - 1}{\mathrm{e}^u - 1}(r_2 - r_1) + r_1 \tag{44}$$

satisfies $r(t'_1; u) = r_1$, $r(t'_2; u) = r_2$ and $r(t'_3; u) = r_3$, provided that u is chosen as the root of

$$h(u) = r(t'_3; u) - r_3 = 0 \tag{45}$$

6.3.4 Estimating the inflation rate

With Insight 1 in mind, we simply choose some constant value for the inflation in the range 0% to 5%.

6.3.5 Estimating future PE ratios

Equipped with the analytical expressions (41) and (44), and an assumed constant $\alpha(t) = \alpha$, (39) becomes

$$\bar{w}^E(t_{K+k}; t_K) = \frac{\displaystyle\prod_{i=1}^{k}\left(1 + \frac{r(t_{K+i}; u)\Delta t}{\rho(t_{K+i})}\right)\frac{e(t_{K+k}; s)\rho(t_{K+k})}{e_K\rho_K}}{(1 + \alpha\Delta t)^k} \geqslant \bar{w}_{\text{min}} \qquad \text{for any } K, k > 0 \tag{46}$$

where $r(t) \equiv d(t)/e(t)$ is an estimate of the dividend payout ratio at time t, and s and u are intensity parameters obtained as the roots of (43) and (45), respectively.

Of what use is (46) to us? We can use it to relate future real equity worths to expected future expensiveness of equities relative to that of bonds. In particular, knowing what the present PE ratio is for equities, we may use (46) to prescribe lower bounds for future PE ratios in order that future real equity worths exceed future real bond worths. The unknowns in (46) are the $\rho(t_{K+i})$, $i = 1, \ldots, k$. Since the values are ostensibly unknown, we may as well proceed by assuming a linear variation of $\rho(t)$ over the future $(t_K, t_{K+k}]$ time interval as:

$$\rho(t) = \frac{(t - t_K)\rho(t_{K+k}) + (t_{K+k} - t)\rho_K}{t_{K+k} - t_K} \tag{47}$$

Clearly this satisfies $\rho(t_K) = \rho_K$. Substituting into (46) gives

$$\frac{\displaystyle\prod_{i=1}^{k}\left(1 + \frac{(t_{K+k} - t_K) r(t_{K+i}; u)\Delta t}{(t_{K+i} - t_K)\rho(t_{K+k}) + (t_{K+k} - t_{K+i})\rho_K}\right)\frac{e(t_{K+k}; s)}{e_K \rho_K}\rho(t_{K+k})}{(1 + \alpha\Delta t)^k} \geqslant \bar{w}_{\text{min}}$$

From this we obtain a lower bound, $\rho_{\text{min}}(t_{K+k})$, of a future PE ratio ρ at time t_{K+k} while we are stationed at some present time t_K as

$$\rho(t_{K+k}) \geqslant \rho_{\text{min}}(t_{K+k}) \qquad \text{for any } K, k > 0 \tag{48}$$

where $\rho_{\text{min}}(t_{K+k})$ is determined by the non-linear implicit equation

$$\begin{aligned} 0 \;=\; &\prod_{i=1}^{k}\left(1 + \frac{(t_{K+k} - t_K) r(t_{K+i}; u)\Delta t}{(t_{K+i} - t_K)\rho_{\text{min}} + (t_{K+k} - t_{K+i})\rho_K}\right)\rho_{\text{min}} \\ &- (1 + \alpha\Delta t)^k \,\frac{e_K \rho_K}{e(t_{K+k}; s)}\,\bar{w}_{\text{min}} \qquad \text{for any } K, k > 0 \end{aligned} \tag{49}$$

Recall that s and u are the “intensity parameters” specified by (43) and (45), and that we have assumed $\Delta t_l = \Delta t$ for all l. Under that latter assumption, at any time t_{K+i}, $i = 0, \ldots, k$, (47) simplifies to

$$\rho(t_{K+i}) = \frac{i\rho(t_{K+k}) + (k - i)\rho_K}{k} \qquad \text{for any } K, k > 0 \text{ and } i = 0, \ldots, k$$

And so (38), (48) and (49) simplify to

$$\boxed{\begin{aligned} &\bar{w}_{\text{min}} \;>\; \langle \bar{w}^B \rangle \\ &\rho(t_{K+k}) \;\geqslant\; \rho_{\text{min}}(t_{K+k}) \\ &0 \;=\; \prod_{i=1}^{k}\Big(1 + \frac{k\, r(t_{K+i}; u)\Delta t}{i\,\rho_{\text{min}} + (k - i)\rho_K}\Big)\rho_{\text{min}} \\ &\qquad\qquad - \;(1 + \alpha\Delta t)^k \frac{e_K \rho_K}{e(t_{K+k}; s)}\,\bar{w}_{\text{min}} \qquad \text{for any } K, k > 0 \end{aligned}} \tag{50}$$

Unfortunately, for values of $k > 2$, obtaining a closed-form analytic solution for $\rho_{\text{min}}(t_{K+k})$ is not trivial. So we shall resort to numerical solutions, again using the Bisection Method. But for interest sake, consider the special case of $k = 1$. Then (50) reduces to

$$\rho(t_{K+1}) \;\geqslant\; \rho_{\text{min}}(t_{K+1}) \;=\; (1 + \alpha\Delta t)\frac{e_K \rho_K}{e(t_{K+1}; s)}\,\bar{w}_{\text{min}} - r(t_{K+1}; u)\Delta t \qquad \text{for any } K > 0$$

A computed history of $\rho_{\text{min}}(t_{K+10})$, i.e., with $k = 10$, for the US S&P 500 index is shown in Figure 16. To arrive at a value $\bar{w}_{\text{min}} = 1.8$, we demanded the *relative real equity worth premium* above that of bonds to equal 0.6. The long-term time average of the relative real bond worth was calculated from the raw data to be $\langle \bar{w}^B \rangle = 1.2$. Also see Figure 14 on page 27.

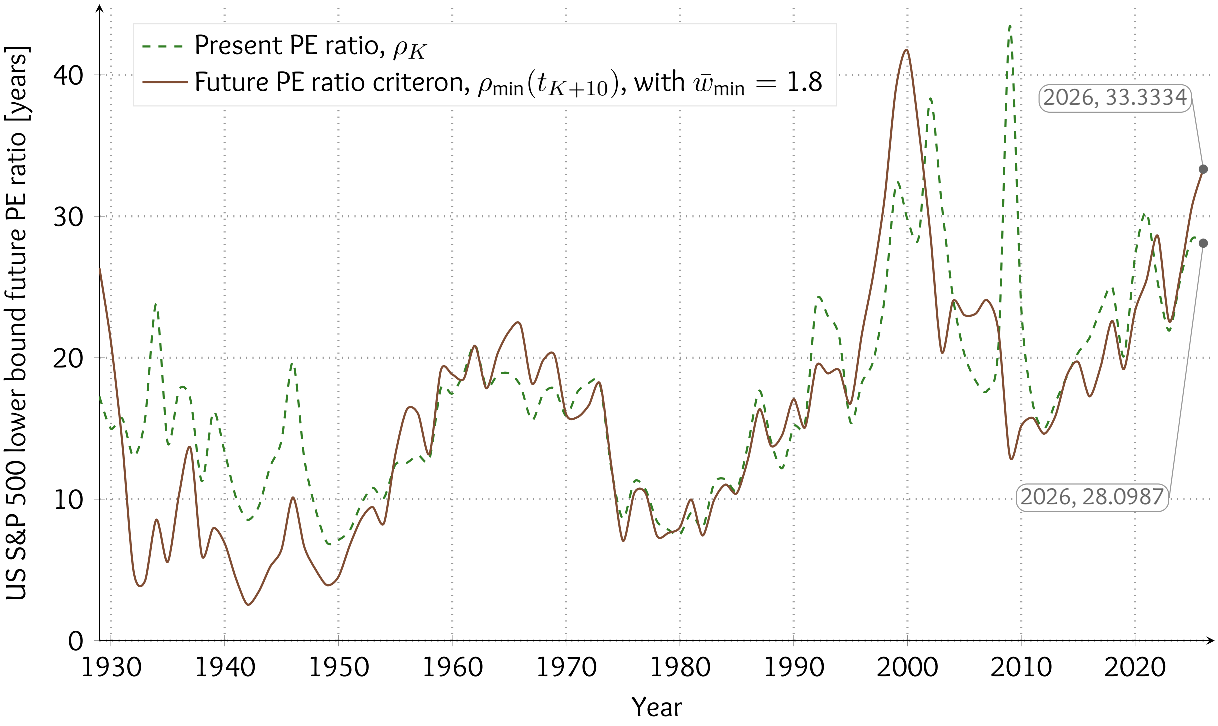

Figure 16: History of the US S&P 500 future PE ratio criterion, $\rho_{\text{min}}(t_{K+10})$, with $k = 10$. What does the criterion stipulate? That if the investor wishes to obtain a *relative real equity worth* greater than or equal to $\bar{w}_{\text{min}} = 1.8$ over the $(t_K, t_{K+10}]$ time interval, i.e., 10 years from some present time t_K, then the actual future PE ratio $\rho(t_{K+10})$ must be greater than $\rho_{\text{min}}(t_{K+10})$. The criterion is given by (50). The viability of the criterion depends on estimating values for the future earning rate per share $e(t)$ and the dividend payout ratio $r(t)$ over the $(t_K, t_{K+10}]$ time interval. They were both estimated with exponential functional forms as per (41) on page 32 and (44) on page 32, respectively. The inflation rate $\alpha(t)$ over that same future time interval was assumed to be constant at 3.15% per year, which is the long-term average (Figure 1 on page 8). The history of the actual S&P 500 present PE ratio ρ_K is also shown. Its long-term time average is about 17.5 years. (Raw data sources: `www.multpl.com`,[4, 9] Federal Reserve Bank of St Louis,[1] Yale University,[2] S&P Global,[5] Macrotrends,[7] Guru Focus,[8] Yale University.[6])

According to the model contemplated in (50), if we can solve for $\rho_{\text{min}}(t_{K+k})$ while stationed at some present time t_K, and if the investor wishes to obtain a relative real equity worth greater than or equal to $\bar{w}_{\text{min}}$ over the $(t_K, t_{K+k}]$ time interval, then the actual future PE ratio $\rho(t_{K+k})$ must not be less than $\rho_{\text{min}}(t_{K+k})$. It is clear, then, that estimating $\rho_{\text{min}}(t_{K+k})$ is important in signalling whether or not to invest in equities at the present time t_K. Since we are unable to estimate a future $\rho(t_{K+k})$, how may we at least estimate the *likelihood* that $\rho(t_{K+k})$ will satisfy (50)? By appealing to the long-term time average of ρ, namely, $\langle\rho\rangle$. Because expensive equity markets will always become cheaper in the future, and cheap markets will always become more expensive, the notion of such a long-term time average value $\langle\rho\rangle$ of expensiveness is sensible. It is reasonable to contemplate ρ_{min} in relation to such an average. And so I propose the following simple forward-looking ruleset to signal when to buy, hold or sell equities at the present time t_K.

Algorithm 3 Strategy 3—Signalling based on future equity expensiveness

```
For any time $t_K$:
    $\langle\rho\rangle = 17.5$ years
    $\lambda = 0.1$
    if $\rho_{\text{min}}(t_{K+k}) < (1-\lambda)\langle\rho\rangle$ then
        Buy equity
    else if $\rho_{\text{min}}(t_{K+k}) \leqslant (1+\lambda)\langle\rho\rangle$ then
        Hold equity
    else
        Sell equity
```

In other words, if the model anticipates a *low* lower bound on the future PE ratio relative to a long-term average, then the signal is to buy equities now. But if the model anticipates a lower bound that is close to that average, then the signal is to hold equities now. Otherwise, the signal is to sell equities now.

6.4 Strategy 4—Signalling based on present-day and future equity expensiveness

The third signalling strategy is an experimental one. It compares both present-day and future equity expensiveness to a long-term average. The following ruleset implements this strategy.

Algorithm 4 Strategy 4—Signalling based on present-day and future equity expensiveness

```
For any time $t_K$:
    $\langle\rho\rangle = 17.5$ years
    $\lambda = 0.1$
    if $\left(\rho_{\text{min}}(t_{K+10}) \leqslant (1-\lambda)\langle\rho\rangle\right)$ or $\left(\rho_K \leqslant \langle\rho\rangle\right)$ or $\left(\rho_{\text{min}}(t_{K+10}) \leqslant (1+\lambda)\langle\rho\rangle$ and $\rho_K \leqslant (1+\lambda)\langle\rho\rangle\right)$
    then
        Buy equity
    else if $\left(\rho_{\text{min}}(t_{K+10}) \leqslant (1+\lambda)\langle\rho\rangle$ and $\rho_K > (1+\lambda)\langle\rho\rangle\right)$ then
        Hold equity
    else
        Sell equity
```

6.5 Put it to the test

How can we assess the viability of the model and the abovementioned signalling rulesets? By applying the model and the rulesets at many historic moments for which data are available. In preparing to do so, the following simple algorithm was used to calibrate the success or failure of the buy, hold and sell signals.

Algorithm 5 Calibrating the success or failure of an equity signalling ruleset

```
For any past time $t_K$ such that time $t_{K+10}$ is also in the past:
  $\langle \bar{w}^B \rangle = 1.2$
  $\bar{w}_{\text{min}} = \langle \bar{w}^B \rangle + 0.6 = 1.8$
  $\mu = 0.1$
  if Buy equity signal then
    if $\bar{w}^E_{K+10,K} \geqslant (1+\mu)\bar{w}_{\text{min}}$ then
      Signal good ☺
    else if $\bar{w}^E_{K+10,K} \geqslant (1-\mu)\bar{w}_{\text{min}}$ then
      Signal neutral 😐
    else
      Signal bad ☹
  else if Hold equity signal then
    if $\bar{w}^E_{K+10,K} \geqslant (1+\mu)\bar{w}_{\text{min}}$ then
      Signal neutral 😐
    else if $\bar{w}^E_{K+10,K} \geqslant (1-\mu)\bar{w}_{\text{min}}$ then
      Signal good ☺
    else
      Signal bad ☹
  else if Sell equity signal then
    if $\bar{w}^E_{K+10,K} < (1-\mu)\bar{w}_{\text{min}}$ then
      Signal good ☺
    else
      Signal bad ☹
```

In the following table, equity buy, hold and sell signals of the US S&P 500 index were calculated using the third signalling ruleset (Algorithm 3 on page 35). For each present date t_K in the table, the ensuing investment period was assumed to be 10 years, starting at t_K. In the table, the 'ρ_K' column lists known historical values of the PE ratio at each t_K date (See Figure 7 on page 13). For each date, a lower bound $\rho_{\text{min}}(t_{K+10})$ on the future PE ratio was calculated using the model in (50) on page 33 (See also Figure 16 on page 34). The '$\langle \bar{w}^E \rangle$' column lists the long-term time average of the relative real equity worth $\bar{w}^E_{K+10,K}$. The '$\bar{w}^E_{K+10,K}$' column lists that actual relative real equity worth that was obtained after investing in equities at the date t_K, ending 10 years hence at the date t_{K+10}, and with dividends reinvested (See Figure 10 on page 18). The '$\bar{w}_{\text{min}}$' column lists a stipulated minimum value of the relative real equity worth. It was calculated as $\bar{w}_{\text{min}} = \langle \bar{w}^B \rangle + 0.6 = 1.2 + 0.6 = 1.8$, where $\langle \bar{w}^B \rangle$ is the long-term average of the relative real bond worth, with the investor's receipt of bond coupons reinvested (See Figure 14 on page 27 and Figure 15 on page 29). Finally, the 'Signal verdict' column lists the outcome of the abovementioned signalling ruleset, as a success (☺), neutral (😐), or a failure (☹). I think this ruleset's overall signal reliability score as shown in the footer of the table is noteworthy.

Date	*PE ratio [years]*			*Signal*	*Relative real equity worth*			*Signal verdict*
t_K	$\langle \rho \rangle$	ρ_K	$\rho_{\text{min}}(t_{K+10})$		$\langle \bar{w}^E \rangle$	$\bar{w}_{\text{min}}$	$\bar{w}^E_{K+10,K}$	
1929-01-01	17.5	17.3	26.3	Sell	2.2	1.8	1.0	☺
1930-01-01	17.5	15.0	21.2	Sell	2.2	1.8	1.2	☺
1931-01-01	17.5	15.7	14.0	Buy	2.2	1.8	1.3	☹
1932-01-01	17.5	13.0	4.9	Buy	2.2	1.8	1.7	😐
1933-01-01	17.5	15.6	4.2	Buy	2.2	1.8	1.8	😐
1934-01-01	17.5	23.8	8.6	Buy	2.2	1.8	1.4	☹
1935-01-01	17.5	14.0	5.6	Buy	2.2	1.8	1.9	😐

To next page.

From previous page.

Date	PE ratio [years]			Signal	Relative real equity worth			Signal verdict
t_K	$\langle\rho\rangle$	ρ_K	$\rho_{\text{min}}(t_{K+10})$		$\langle\bar{w}^E\rangle$	$\bar{w}_{\text{min}}$	$\bar{w}^E_{K+10,K}$	
1936-01-01	17.5	17.7	10.4	Buy	2.2	1.8	1.7	😐
1937-01-01	17.5	17.0	13.6	Buy	2.2	1.8	1.0	☹
1938-01-01	17.5	11.3	6.0	Buy	2.2	1.8	1.3	☹
1939-01-01	17.5	16.2	7.9	Buy	2.2	1.8	1.2	☹
1940-01-01	17.5	13.3	6.9	Buy	2.2	1.8	1.4	☹
1941-01-01	17.5	10.1	4.4	Buy	2.2	1.8	2.0	😐
1942-01-01	17.5	8.5	2.5	Buy	2.2	1.8	2.8	☺
1943-01-01	17.5	9.5	3.5	Buy	2.2	1.8	2.8	☺
1944-01-01	17.5	12.3	5.2	Buy	2.2	1.8	2.4	☺
1945-01-01	17.5	14.2	6.5	Buy	2.2	1.8	3.0	☺
1946-01-01	17.5	19.7	10.1	Buy	2.2	1.8	2.9	☺
1947-01-01	17.5	12.9	6.6	Buy	2.2	1.8	4.0	☺
1948-01-01	17.5	9.3	5.0	Buy	2.2	1.8	3.9	☺
1949-01-01	17.5	6.9	3.9	Buy	2.2	1.8	4.9	☺
1950-01-01	17.5	7.1	4.5	Buy	2.2	1.8	4.3	☺
1951-01-01	17.5	7.8	6.7	Buy	2.2	1.8	3.7	☺
1952-01-01	17.5	9.7	8.6	Buy	2.2	1.8	3.7	☺
1953-01-01	17.5	10.8	9.4	Buy	2.2	1.8	3.2	☺
1954-01-01	17.5	10.1	8.3	Buy	2.2	1.8	3.7	☺
1955-01-01	17.5	12.4	13.1	Buy	2.2	1.8	2.9	☺
1956-01-01	17.5	12.6	16.3	Hold	2.2	1.8	2.5	😐
1957-01-01	17.5	13.1	16.0	Hold	2.2	1.8	2.1	😐
1958-01-01	17.5	12.8	13.2	Buy	2.2	1.8	2.6	☺
1959-01-01	17.5	17.8	19.1	Hold	2.2	1.8	2.0	😐
1960-01-01	17.5	17.5	18.8	Hold	2.2	1.8	1.6	☺
1961-01-01	17.5	18.9	18.5	Hold	2.2	1.8	1.6	☹
1962-01-01	17.5	20.9	20.9	Sell	2.2	1.8	1.5	☺
1963-01-01	17.5	17.9	17.8	Hold	2.2	1.8	1.8	☺
1964-01-01	17.5	18.8	20.3	Sell	2.2	1.8	1.1	☺
1965-01-01	17.5	18.9	21.9	Sell	2.2	1.8	0.7	☺
1966-01-01	17.5	18.0	22.3	Sell	2.2	1.8	0.8	☺
1967-01-01	17.5	15.6	18.2	Hold	2.2	1.8	1.0	☹
1968-01-01	17.5	17.5	19.8	Sell	2.2	1.8	0.7	☺
1969-01-01	17.5	17.8	20.2	Sell	2.2	1.8	0.7	☺
1970-01-01	17.5	15.9	16.0	Hold	2.2	1.8	0.9	☹
1971-01-01	17.5	17.6	15.8	Hold	2.2	1.8	1.0	☹
1972-01-01	17.5	18.2	16.6	Hold	2.2	1.8	0.8	☹
1973-01-01	17.5	18.0	18.1	Hold	2.2	1.8	0.8	☹
1974-01-01	17.5	12.0	12.2	Buy	2.2	1.8	1.3	☹
1975-01-01	17.5	8.6	7.1	Buy	2.2	1.8	1.8	😐
1976-01-01	17.5	11.3	10.4	Buy	2.2	1.8	1.7	😐
1977-01-01	17.5	10.7	10.4	Buy	2.2	1.8	2.1	☺
1978-01-01	17.5	8.4	7.4	Buy	2.2	1.8	2.3	☺

To next page.

From previous page.

Date	PE ratio [years]			Signal	Relative real equity worth			Signal verdict
t_K	$\langle\rho\rangle$	ρ_K	$\rho_{\min}(t_{K+10})$		$\langle\bar{w}^E\rangle$	$\bar{w}_{\min}$	$\bar{w}^E_{K+10,K}$	
1979-01-01	17.5	7.8	7.6	Buy	2.2	1.8	2.5	☺
1980-01-01	17.5	7.5	8.0	Buy	2.2	1.8	2.8	☺
1981-01-01	17.5	9.0	10.0	Buy	2.2	1.8	2.3	☺
1982-01-01	17.5	8.0	7.4	Buy	2.2	1.8	3.5	☺
1983-01-01	17.5	11.1	9.9	Buy	2.2	1.8	2.9	☺
1984-01-01	17.5	11.4	11.0	Buy	2.2	1.8	2.8	☺
1985-01-01	17.5	10.6	10.4	Buy	2.2	1.8	2.6	☺
1986-01-01	17.5	13.9	12.9	Buy	2.2	1.8	2.8	☺
1987-01-01	17.5	17.7	16.4	Hold	2.2	1.8	2.7	😐
1988-01-01	17.5	14.1	13.8	Buy	2.2	1.8	3.6	☺
1989-01-01	17.5	12.2	14.6	Buy	2.2	1.8	4.1	☺
1990-01-01	17.5	15.1	17.1	Hold	2.2	1.8	4.0	😐
1991-01-01	17.5	15.4	15.1	Buy	2.2	1.8	3.9	☺
1992-01-01	17.5	24.0	19.4	Sell	2.2	1.8	2.6	☹
1993-01-01	17.5	23.0	18.9	Hold	2.2	1.8	1.9	☺
1994-01-01	17.5	21.5	19.1	Hold	2.2	1.8	2.2	😐
1995-01-01	17.5	15.4	16.8	Hold	2.2	1.8	2.3	😐
1996-01-01	17.5	18.2	21.7	Sell	2.2	1.8	1.9	☹
1997-01-01	17.5	19.8	25.9	Sell	2.2	1.8	1.7	☹
1998-01-01	17.5	24.3	31.3	Sell	2.2	1.8	1.3	☺
1999-01-01	17.5	32.3	39.1	Sell	2.2	1.8	0.6	☺
2000-01-01	17.5	29.8	41.7	Sell	2.2	1.8	0.7	☺
2001-01-01	17.5	28.5	36.1	Sell	2.2	1.8	0.9	☺
2002-01-01	17.5	38.3	28.8	Sell	2.2	1.8	1.1	☺
2003-01-01	17.5	30.9	20.4	Sell	2.2	1.8	1.6	☺
2004-01-01	17.5	24.3	24.0	Sell	2.2	1.8	1.6	☺
2005-01-01	17.5	20.3	23.0	Sell	2.2	1.8	1.7	☹
2006-01-01	17.5	18.4	23.1	Sell	2.2	1.8	1.5	☺
2007-01-01	17.5	17.7	24.1	Sell	2.2	1.8	1.6	☹
2008-01-01	17.5	20.9	21.8	Sell	2.2	1.8	2.1	☹
2009-01-01	17.5	43.5	13.0	Buy	2.2	1.8	3.1	☺
2010-01-01	17.5	23.5	15.2	Buy	2.2	1.8	3.0	☺
2011-01-01	17.5	16.8	15.7	Buy	2.2	1.8	3.0	☺
2012-01-01	17.5	15.0	14.6	Buy	2.2	1.8	3.4	☺
2013-01-01	17.5	16.8	15.9	Hold	2.2	1.8	2.5	😐
2014-01-01	17.5	18.7	18.7	Hold	2.2	1.8	2.4	😐
2015-01-01	17.5	20.4	19.7	Sell	2.2	1.8	2.6	☹
2016-01-01	17.5	21.4	17.3	Hold	2.2	1.8	3.1	😐
2017-01-01	17.5	23.5	19.4	Sell	2.2	1.8		
2018-01-01	17.5	25.0	22.6	Sell	2.2	1.8		
2019-01-01	17.5	20.1	19.2	Hold	2.2	1.8		
2020-01-01	17.5	27.1	23.3	Sell	2.2	1.8		
2021-01-01	17.5	30.3	25.5	Sell	2.2	1.8		
2022-01-01	17.5	25.3	28.6	Sell	2.2	1.8		

To next page.

From previous page.

Date	*PE ratio* [years]			*Signal*	*Relative real equity worth*			*Signal*
t_K	$\langle\rho\rangle$	ρ_K	$\rho_{\text{min}}(t_{K+10})$		$\langle\bar{w}^E\rangle$	$\bar{w}_{\text{min}}$	$\bar{w}^E_{K+10,K}$	*verdict*
2023-01-01	17.5	21.9	22.6	Sell	2.2	1.8		
2024-01-01	17.5	25.6	26.0	Sell	2.2	1.8		
2025-01-01	17.5	28.4	30.7	Sell	2.2	1.8		
2026-01-01	17.5	28.1	33.3	Sell	2.2	1.8		
				Signal verdicts: 51 🙂, 17 😐, 20 ☹.		*Signal reliability:*		0.77

End.

By applying the four different equity signalling rulesets to the time series data spanning nearly the last 100 years, it is clear that Strategy 3 is favoured because its signal reliability score is the highest.

	Signal reliability
Strategy 1—No signalling, so "Buy and Hold"	0.67
Strategy 2—Signalling based only on present-day equity expensiveness	0.65
Strategy 3—Signalling based on future equity expensiveness	0.77
Strategy 4—Signalling based on present-day and future equity expensiveness	0.74

Adopting Strategy 3, then, the sensitivity of its signalling ruleset to various input parameters was analysed, as shown in the following table. As expected, the signal reliability rises as the period of investment widens, with 10 years being optimal. The signal reliability also rises as the prescribed lower bound $\bar{w}_{\text{min}}$ on the future relative real equity worth falls. This is also expected because as this prescribed lower bound falls, it means that the investor is more willing to accept a reduced future equity worth over the investment period, which in turn reduces the number of actual real equity worths over time that would trigger a sell signal.

Strategy 3—Sensitivity of signal reliability to various input parameters

Investment period k [years]	*Spread factor on* $\langle\rho\rangle$ λ	*Spread factor on* $\langle\bar{w}^E\rangle$ μ	$\bar{w}_{\text{min}}$	*Date for* $e(t_3)$ t_3	*Signal reliability*
10	0.1	0.1	1.8	2025-06-30	0.77
2					0.68
5					0.60
15					0.76
10	0.05				0.73
	0.01				0.74
	0.1	0.05			0.76
		0.01			0.75
		0.1	2.2		0.68
			2.0		0.74
			1.6		0.75
			1.4		0.82
			1.2		0.85
			1.0		0.92
			1.8	2025-09-30	0.76
				2026-03-31	0.77

In summary, the question of when equities can be considered an attractive investment was answered in this section. Four different investment strategies were identified and applied to the historical time series data. A novel long-term investment strategy was introduced here as Strategy 3, and its viability ratified. It is a forward-looking strategy. It was shown to have the highest signalling reliability score at 0.77. Its ruleset is based on a quantitative model in which certain unknown future variables are extrapolated from the historical data whilst appealling to various key insights obtained from the historical data. The ruleset offers buy, hold and sell signals at any moment. I find it encouraging that in an ever-changing investment landscape, the model, together with its novel signalling ruleset, seems to offer a mid- to long-term investor a capability with which to make sensible investment decisions.

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