A Circular Variable Distance Measure

Paul Kotschy

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A FORMAL ANALYSIS of circular variables is given. Sensible distance measures are derived for when a circular variable is continous and for when it is discrete.

Continuous circular variables. The analysis begins by considering the simple real line. Each point on the real line is uniquely assigned a single real number, so that two different numbers denote two different points, as shown in Figure 1. The calculation of the distance between any two such points is, trivially, the difference between their two assigned real numbers.

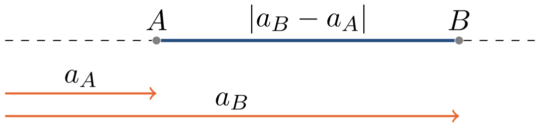

Figure 1: Two points A and B are located at the positions a_A and a_B on the real line. Their separation is, trivially, $|a_B - a_A|$.

But if the real line is wrapped around a circular cylinder, say, then the unique assignment fails because there are infinitely many numbers each referring to the same point. That is, one number is assigned for each wrapping instance. And the calculation of a distance measure between two such points is therefore no longer as trivial.

Consider two points A and B located at the angular positions α_A and α_B on the circumference of a circle of radius r, as shown in Figure 2. The two points' smallest circumferential separation, $S(\alpha_A, \alpha_B, r)$, is

$$S(\alpha_A, \alpha_B, r) = \min(|\alpha_B - \alpha_A|, 2\pi - |\alpha_B - \alpha_A|)r$$

The use of functional notation for "$S(\alpha_A, \alpha_B, r)$" indicates explicitly the dependence of the circumferential separation S on three variables, namely, A's angular position α_A, B's angular position α_B, and the radius r of the circle. The 'min' function accepts two arguments, returning the numerically lesser of the two. And conventionally, the angular positions α_A and α_B are measured in radians, so that a single angular revolution equals 2π radians.

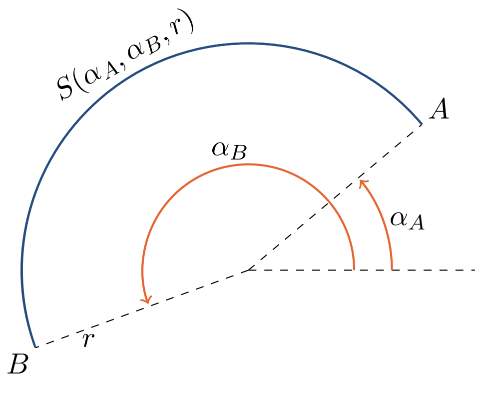

Figure 2: Two points A and B are located at the angular positions α_A and α_B on the circumference of a circle of radius r. Their circumferential separation is denoted $S(\alpha_A, \alpha_B, r)$.

The point circumferentially furthest from A is located at an angle $\alpha_A + \pi$. Its circumferential separation from A is therefore

$$S(\alpha_A, \alpha_A + \pi, r) = \min(\pi, 2\pi - \pi)r = \pi r$$

as expected.

However, while geometrically, a furthest distance equal to πr is sensible, it may seem arbitrary for a variable for which no obvious geometric scale is apparent. Specifically, the choice of value for the radius r is arbitrary. We therefore define a "normalized" separation D by setting $r = 1/\pi$ so that the furthest point from A is at a distance 1 from A. That is

$$\begin{aligned} D(\alpha_A, \alpha_B) &= \frac{1}{\pi r} S(\alpha_A, \alpha_B, r) \\ &= \frac{1}{\pi} \min(|\alpha_B - \alpha_A|, 2\pi - |\alpha_B - \alpha_A|) \end{aligned} \tag{1}$$

Thus $D(\alpha_A, \alpha_A + \pi) = 1$ for all α_A.

The definition of separation $D(\alpha_A, \alpha_B)$ suffices as a normalised distance measure between points whose angular position assignments are constrained to lie in the interval $[0, 2\pi)$. But this is restrictive. For example, for a variable used to store the day of the week, an appropriate interval is $[1, 8)$. And for a variable used to store the month of the year, an appropriate interval is $[1, 366)$ (or $[1, 367)$ to account for leap years). It is therefore necessary to relax the constraints that $0 \leq \alpha_A < 2\pi$ and $0 \leq \alpha_B < 2\pi$.

Consider the linear transformation:

$$\alpha(\beta) = \left(\frac{\beta - \beta_{\text{lo}}}{\beta_{\text{hi}} - \beta_{\text{lo}}} \right) 2\pi \tag{2}$$

for some arbitrary β_{lo} and β_{hi}. It is clear that when $\beta = \beta_{\text{lo}}$, $\alpha = 0$. And when $\beta = \beta_{\text{hi}}$, $\alpha = 2\pi$. Thus, by considering the α_A and α_B in (1) as functions of β, namely, $\alpha_A = \alpha(\beta_A)$ and $\alpha_B = \alpha(\beta_B)$, and provided that we ensure that the circular variable value lies in the interval $[\beta_{\text{lo}}, \beta_{\text{hi}})$, we may happily shift our attention from α as being the circular variable to β, as shown in Figure 3.

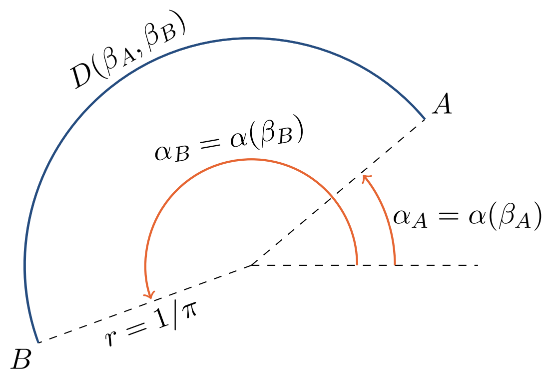

Figure 3: Two points A and B are located at the angular positions $\alpha(\beta_A)$ and $\alpha(\beta_B)$ on the circumference of a circle of radius $r = 1/\pi$, with $\alpha(\beta)$ given by (2). Their normalised circumferential separation $D(\beta_A, \beta_B)$ is given by (3).

Substituting (2) in (1) gives

$$D(\beta_A, \beta_B) = \frac{2}{\beta_{\text{hi}} - \beta_{\text{lo}}} \min(|\beta_A - \beta_B|, \beta_{\text{hi}} - \beta_{\text{lo}} - |\beta_A - \beta_B|) \tag{3}$$

with the functional notation "$D(\beta_A, \beta_B)$" now indicating explicit dependence of D on β_A and β_B instead of on α_A and α_B.

$D(\beta_A, \beta_B)$ is defined using the 'min' and absolute value functions. It may alternatively be defined with the absolute value function only. To achieve this we rely on the easily verifiable result that for any real a and b,

$$\min(a, b-a) \;=\; \frac{b}{2} - \left| a - \frac{b}{2} \right|$$

Recognising that (3) contains the form $\min(a, b-a)$, we obtain after some algebraic manipulation

$$D(\beta_A, \beta_B) = 1 - \left| 1 - \frac{2}{\beta_{\text{hi}} - \beta_{\text{lo}}} \left| \beta_B - \beta_A \right| \right| \tag{4}$$

How may we assess the validity of this result? Firstly, consider any given value β_A for the circular variable β. We expect that the value "furthest" from β_A to be $\beta_A + \frac{1}{2}(\beta_{\text{hi}} - \beta_{\text{lo}})$. We also expect the separation between the values β_{lo} and β_{hi} to vanish. Using (4)

$$D(\beta_A, \beta_A + \frac{1}{2}(\beta_{\text{hi}} - \beta_{\text{lo}})) \;=\; 1 - \left| 1 - \frac{2}{\beta_{\text{hi}} - \beta_{\text{lo}}} \left| \frac{\beta_{\text{hi}} - \beta_{\text{lo}}}{2} \right| \right| \;=\; 1 \;\text{ for any } \beta_A$$

And

$$D(\beta_{\text{lo}}, \beta_{\text{hi}}) \;=\; 1 - \left| 1 - \frac{2}{\beta_{\text{hi}} - \beta_{\text{lo}}} \left| \beta_{\text{hi}} - \beta_{\text{lo}} \right| \right| \;=\; 0$$

Discrete circular variables. Suppose that instead of our circular variable β being continuous over the interval $[\beta_{\text{lo}}, \beta_{\text{hi}})$, its range of values is restricted to a finite set of M evenly spaced discrete values. That is, suppose $\beta = \beta_m$, $m = 1, 2, \ldots, M$. The circularity of β is represented in Figure (4).

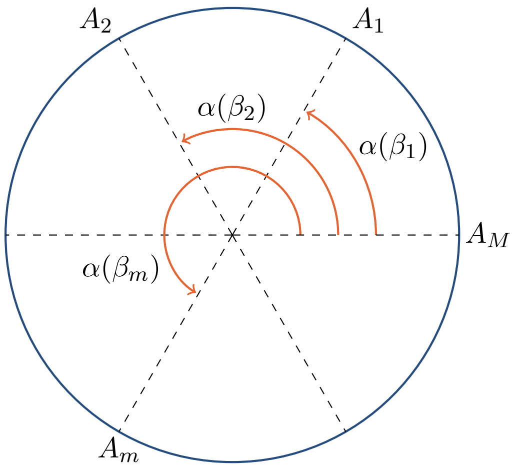

Figure 4: Points $A_1, \ldots, A_M$ are located at the angular positions $\alpha(\beta_1), \ldots, \alpha(\beta_M)$ on the circumference of a circle of radius $r = 1/\pi$, with $\alpha(\beta_m)$ given by (2).

With β now discrete, what is the meaning of β_{hi}? To be sure, $\beta_{\text{hi}} \neq \beta_M$, because if it was, then the distance separation of the final "sector" between A_M and A_1 (Figure (4)) would be lost when moving from a continuous distance measure (4) to a discrete one. Instead, β_{hi} must be thought of as "lying alongside" β_1, so that:

$$\beta_{\text{hi}} = \beta_{\text{lo}} + M(\beta_2 - \beta_1)$$

Substituting into (4) gives

$$D(\beta_A, \beta_B) = 1 - \left| 1 - \frac{2}{M(\beta_2 - \beta_1)} \left| \beta_B - \beta_A \right| \right| \tag{5}$$

For example, consider a discrete circular variable having 5 allowable values ($M = 5$), as represented in Figure 5. We expect the point "furthest" from the discrete point A_1 (with circular variable $\beta = \beta_1$) to be located midway between points A_3 and A_4. The value of the circular variable corresponding to that midway point is $\beta_3 + \frac{1}{2}(\beta_2 - \beta_1)$, notwithstanding that in this example, such a point is inadmissable. Then using (5) to calculate their separation,

$$\begin{aligned} D(\beta_1, \beta_3 + \frac{1}{2}(\beta_2 - \beta_1)) &= 1 - \left|1 - \frac{2}{5(\beta_2 - \beta_1)}\left|\beta_3 + \frac{1}{2}(\beta_2 - \beta_1) - \beta_1\right|\right| \\ &= 1 - \left|1 - \frac{2}{5(\beta_2 - \beta_1)}\left|\beta_1 + 2(\beta_2 - \beta_1) + \frac{1}{2}(\beta_2 - \beta_1) - \beta_1\right|\right| \\ &= 1 \end{aligned} \tag{6}$$

as expected. For any set $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ of 5 evenly spaced values, the matrix of separation values may easily be calculated using (5) as:

	β_1	β_2	β_3	β_4	β_5
β_1	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{2}{5}$
β_2	$\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
β_3	$\frac{4}{5}$	$\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{4}{5}$
β_4	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	0	$\frac{2}{5}$
β_5	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	0

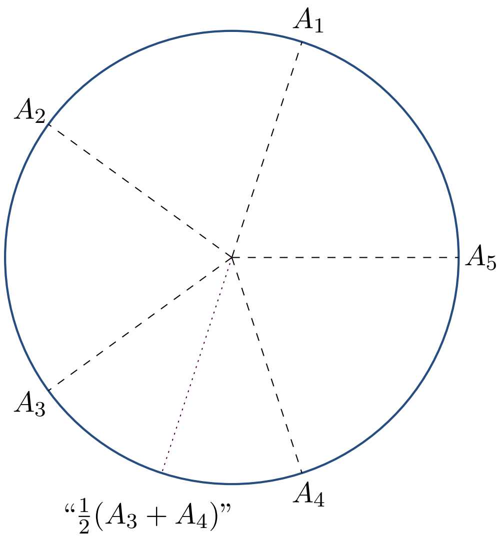

Figure 5: Points $A_1, \ldots, A_5$ are located at the angular positions $\alpha(\beta_1), \ldots, \alpha(\beta_5)$ on the circumference of a circle of radius $r = 1/\pi$, with $\alpha(\beta_m)$ given by (2).

In summary, combining the continuous (4) and the discrete distance measure (5) gives

$$D(\beta_A, \beta_B) = \begin{cases} 1 - \left|1 - \dfrac{2}{\beta_{\text{hi}} - \beta_{\text{lo}}}\left|\beta_B - \beta_A\right|\right|, & \beta_A \text{ and } \beta_B \text{ continuous.} \\ \\ 1 - \left|1 - \dfrac{2}{M(\beta_2 - \beta_1)}\left|\beta_B - \beta_A\right|\right|, & \beta_A \text{ and } \beta_B \text{ discrete. } \beta_A, \beta_B = 1 \ldots M. \end{cases} \tag{7}$$