Rotational Transformations in Three Dimensions

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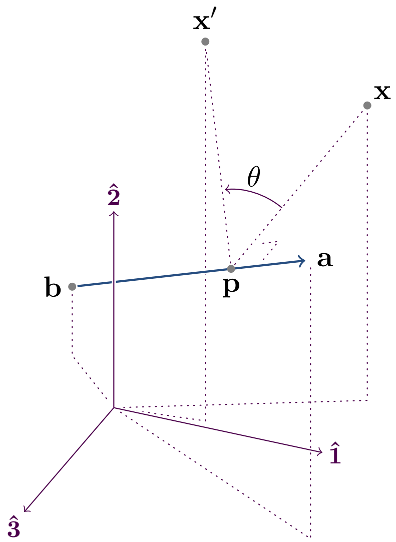


Abstract

In this study I take a tour of rotational transformations in three-dimensions. I begin with the well-known formulae for performing rotations in two dimensions. I then generalise to three dimensions, deriving a useful coordinate-free formula.

Contents

1 Two-dimensional rotations

The description of the rotation of any point on the two-dimensional Euclidean plane is well-known. Such a point may be represented by the position vector

$$\mathbf{x}(\hat{\mathbf{1}}, \hat{\mathbf{2}}) = x\hat{\mathbf{1}} + y\hat{\mathbf{2}} + 0\hat{\mathbf{3}}$$

Writing the point in this form emphasises that the Euclidean plane E^2 in two-dimensions is the set of all linear combinations of the $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ basis vectors:

$$E^2 = \{x\hat{\mathbf{1}} + y\hat{\mathbf{2}} \mid x = \mathbf{x} \cdot \hat{\mathbf{1}}, y = \mathbf{x} \cdot \hat{\mathbf{2}} \in \mathbb{R};\ \hat{\mathbf{1}} \cdot \hat{\mathbf{1}} = \hat{\mathbf{2}} \cdot \hat{\mathbf{2}} = 1,\ \hat{\mathbf{1}} \cdot \hat{\mathbf{2}} = \hat{\mathbf{2}} \cdot \hat{\mathbf{1}} = 0\}$$

The set $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}\}$ is called an *orthonormal basis* of E^2.

It it common practice to represent such vectors with a uni-columnar matrix:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

But I prefer not to because doing so assumes that the $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ basis vectors are represented with

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

That is

$$\mathbf{x}(\hat{\mathbf{1}}, \hat{\mathbf{2}}) = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

I think it preferable to be explicit about which vector basis is used.

Rotate a vector about the origin. A rotation of the position vector $\mathbf{x}(\hat{\mathbf{1}}, \hat{\mathbf{2}})$ about the origin through some angle θ is shown schematically in Figure 1. It is evident that

$$x = |\mathbf{x}| \cos\psi, \quad y = |\mathbf{x}| \sin\psi, \quad x' = |\mathbf{x}'| \cos(\theta + \psi), \quad y' = |\mathbf{x}'| \sin(\theta + \psi) \tag{2}$$

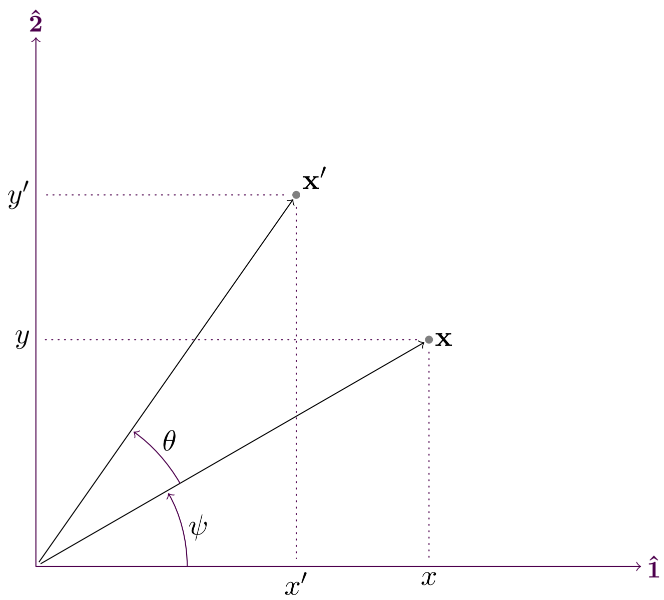


Figure 1: Rotation in two-dimensions about the origin.

A fundamental property of rotation transformations is that they are length preserving. So when $\mathbf{x}$ is rotated onto $\mathbf{x}'$, its distance from the origin does not change. That is, $|\mathbf{x}'| = |\mathbf{x}|$, giving

$$\begin{aligned} x'(x,y,\theta) &= |\mathbf{x}|\,(\cos\theta\cos\psi - \sin\theta\sin\psi) &= \cos\theta\, x - \sin\theta\, y \\ y'(x,y,\theta) &= |\mathbf{x}|\,(\sin\theta\cos\psi + \sin\psi\cos\theta) &= \sin\theta\, x + \cos\theta\, y \end{aligned} \tag{3}$$

The rotation transformation is therefore

$$\boxed{\begin{aligned} R(\theta)\,:\,\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) \mapsto \mathbf{x}'(\theta;\hat{\mathbf{1}},\hat{\mathbf{2}}) &= R(\theta)\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) \\ &= [R(\theta)\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})\cdot\hat{\mathbf{1}}]\,\hat{\mathbf{1}} \;+\; [R(\theta)\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})\cdot\hat{\mathbf{2}}]\,\hat{\mathbf{2}} \\ &= [\mathbf{x}\cdot(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{1}} \;+\; [\mathbf{x}\cdot(\sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{2}} \end{aligned}} \tag{4}$$

If we wish to represent the vector $\mathbf{x}$ in matrix notation, then we may write

$$\begin{bmatrix} x' \\ y' \end{bmatrix}(x,y,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

provided that we represent $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ with (1). We identify the rotation with the well-known matrix

$$R(\theta) \;=\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

Rotate a vector about a point p. To rotate the position vector $\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})$ about some other point $\mathbf{p} = p_1\hat{\mathbf{1}} + p_2\hat{\mathbf{2}}$, other than the origin, as shown in Figure 2, the transformation may be done by first translating $\mathbf{x}$ through $-\mathbf{p}$, rotating through θ, and then translating back through $\mathbf{p}$.

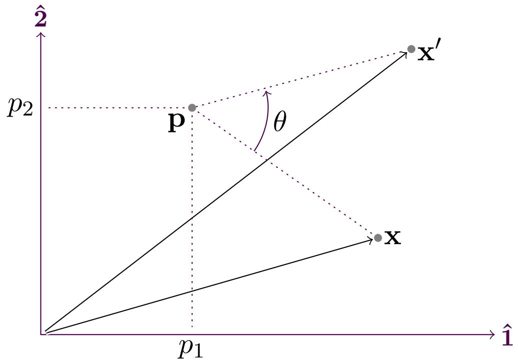


Figure 2: Rotation in two dimensions about an arbitrary point $\mathbf{p}$.

The translation through $\mathbf{p}$ is

$$\boxed{\begin{aligned} T(\mathbf{p})\,:\,\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) \mapsto \mathbf{x}'(\mathbf{p};\hat{\mathbf{1}},\hat{\mathbf{2}}) &= T(\mathbf{p})\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) \\ &= [(\mathbf{x}+\mathbf{p})\cdot\hat{\mathbf{1}}]\,\hat{\mathbf{1}} + [(\mathbf{x}+\mathbf{p})\cdot\hat{\mathbf{2}}]\,\hat{\mathbf{2}} \end{aligned}} \tag{6}$$

It is then easy to verify that the rotation of the position vector $\mathbf{x}$ about $\mathbf{p}$ through angle θ is given by the composite transformation

$$\boxed{\begin{aligned} T(\mathbf{p})R(\theta)T(-\mathbf{p})\,:\,\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) &\mapsto \mathbf{x}'(\theta,\mathbf{p};\hat{\mathbf{1}},\hat{\mathbf{2}}) \\ &= T(\mathbf{p})R(\theta)T(-\mathbf{p})\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}}) \\ &= [(\mathbf{x}-\mathbf{p})\cdot(\cos\theta\hat{\mathbf{1}} - \sin\theta\hat{\mathbf{2}}) + \mathbf{p}\cdot\hat{\mathbf{1}}]\,\hat{\mathbf{1}} \\ &+ [(\mathbf{x}-\mathbf{p})\cdot(\sin\theta\hat{\mathbf{1}} + \cos\theta\hat{\mathbf{2}}) + \mathbf{p}\cdot\hat{\mathbf{2}}]\,\hat{\mathbf{2}} \end{aligned}} \tag{7}$$

Suppose, for example, that $\theta = \dfrac{\pi}{2}$ and $\mathbf{p} = 2\hat{\mathbf{1}} + 1\hat{\mathbf{2}}$, as shown in Figure 3. Then using (7) the point $\mathbf{x} = 3\hat{\mathbf{1}} + 1\hat{\mathbf{2}}$ is rotated to $\mathbf{x}' = 2\hat{\mathbf{1}} + 2\hat{\mathbf{2}}$, and the point $\mathbf{x} = 3\hat{\mathbf{1}} + 0\hat{\mathbf{2}}$ is rotated to $\mathbf{x}' = 3\hat{\mathbf{1}} + 2\hat{\mathbf{2}}$, as expected.

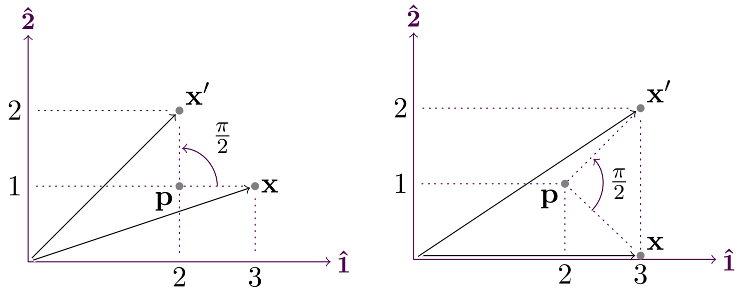


Figure 3: Rotations in two dimensions about point $\mathbf{p} = 2\hat{\mathbf{1}} + 1\hat{\mathbf{2}}$ through angle $\theta = \dfrac{\pi}{2}$.

Rotate the space. We have thus far considered rotating a point by transforming the point's position vector. That is, under the action of the transformation, the position vector changes while the underlying space is held constant. However, an almost equivalent consideration is for the vector to be held constant and for the space to change. I show the equivalence using two approaches. The first approach is an intuitive one.

In (4), the point's position vector $x\hat{\mathbf{1}} + y\hat{\mathbf{2}}$ was transformed to $x'(x, y, \theta)\hat{\mathbf{1}} + y'(x, y, \theta)\hat{\mathbf{2}}$, which is obviously a different vector. But suppose the position vector is held constant, and is simply expressed under a new orthonormal basis set $\{\hat{\mathbf{1}}', \hat{\mathbf{2}}'\}$ of E^2 which has been rotated about the origin through an angle ϕ, say, relative to the $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}\}$ basis. Then we must be able to write

$$\begin{aligned} x\hat{\mathbf{1}} + y\hat{\mathbf{2}} &= \mathbf{x}(\hat{\mathbf{1}}, \hat{\mathbf{2}}) \\ &= \mathbf{x}(\phi; \hat{\mathbf{1}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}}), \hat{\mathbf{2}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}})) \\ &= \bar{x}(x, y, \phi)\hat{\mathbf{1}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}}) + \bar{y}(x, y, \phi)\hat{\mathbf{2}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}}) \end{aligned} \tag{8}$$

for some $\bar{x}(x, y, \phi)$ and $\bar{y}(x, y, \phi)$. The abovementioned equivalence can be established if we can find the conditions under which $\bar{x}(x, y, \phi) = x'(x, y, \theta)$ and $\bar{y}(x, y, \phi) = y'(x, y, \theta)$. A rotation of the $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}\}$ basis set about the origin through angle ϕ onto $\{\hat{\mathbf{1}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}}), \hat{\mathbf{2}}'(\phi; \hat{\mathbf{1}}, \hat{\mathbf{2}})\}$ is shown schematically in Figure 4. Noting that in the diagram, $\phi < 0$, we have

$$\begin{aligned} \hat{\mathbf{1}}' &= \left|\hat{\mathbf{1}}'\right| \cos\phi\,\hat{\mathbf{1}} + \left|\hat{\mathbf{1}}'\right| \sin\phi\,\hat{\mathbf{2}} = \cos\phi\,\hat{\mathbf{1}} + \sin\phi\,\hat{\mathbf{2}} \\ \hat{\mathbf{2}}' &= -\left|\hat{\mathbf{2}}'\right| \sin\phi\,\hat{\mathbf{1}} + \left|\hat{\mathbf{2}}'\right| \cos\phi\,\hat{\mathbf{2}} = -\sin\phi\,\hat{\mathbf{1}} + \cos\phi\,\hat{\mathbf{2}} \end{aligned}$$

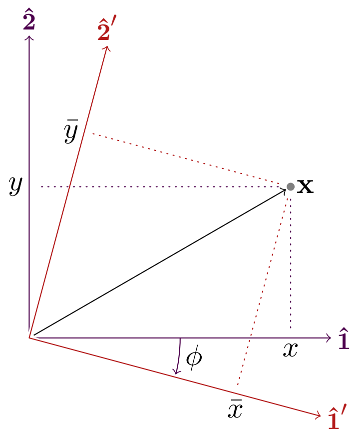


Figure 4: Rotation of the $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ basis vectors in two-dimensions about the origin through angle ϕ, with position vector $\mathbf{x}$ held constant.

Substituting into (8),

$$x\hat{\mathbf{1}} + y\hat{\mathbf{2}} = (\cos\phi\,\bar{x} - \sin\phi\,\bar{y})\hat{\mathbf{1}} + (\sin\phi\,\bar{x} + \cos\phi\,\bar{y})\hat{\mathbf{2}}$$

Since $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ are linearly independent,

$$\begin{aligned}\bar{x}(x,y,\phi) &= \cos(-\phi)\,x - \sin(-\phi)\,y \\ \bar{y}(x,y,\phi) &= \sin(-\phi)\,x + \cos(-\phi)\,y\end{aligned} \tag{9}$$

Comparing (9) with (3), the equivalence is established whenever $\phi = -\theta$. That is, the coordinates of a vector which has been rotated through a given angle θ equals the coordinates of the original vector but which is now expressed in a coordinate system which has been rotated through an angle $-\theta$.

The second approach is algebraic. Consider rotating the $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ basis vectors about the origin through an angle $(-\theta)$. Applying (4), the rotation of $\hat{\mathbf{1}}$ about angle $(-\theta)$ is

$$\begin{aligned}R(-\theta)\hat{\mathbf{1}} &= [\hat{\mathbf{1}}\cdot(\cos(-\theta)\hat{\mathbf{1}} - \sin(-\theta)\hat{\mathbf{2}})]\,\hat{\mathbf{1}} \;+\; [\hat{\mathbf{1}}\cdot(\sin(-\theta)\hat{\mathbf{1}} + \cos(-\theta)\hat{\mathbf{2}})]\,\hat{\mathbf{2}} \\ &= \cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}}\end{aligned}$$

And the rotation of $\hat{\mathbf{2}}$ about $(-\theta)$ is

$$\begin{aligned}R(-\theta)\hat{\mathbf{2}} &= [\hat{\mathbf{2}}\cdot(\cos(-\theta)\hat{\mathbf{1}} - \sin(-\theta)\hat{\mathbf{2}})]\,\hat{\mathbf{1}} \;+\; [\hat{\mathbf{2}}\cdot(\sin(-\theta)\hat{\mathbf{1}} + \cos(-\theta)\hat{\mathbf{2}})]\,\hat{\mathbf{2}} \\ &= \sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}}\end{aligned}$$

Define two new vectors $\hat{\mathbf{1}}' = R(-\theta)\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}' = R(-\theta)\hat{\mathbf{2}}$. Since $\hat{\mathbf{1}}'\cdot\hat{\mathbf{2}}' = 0$ and $\hat{\mathbf{1}}'\cdot\hat{\mathbf{1}}' = \hat{\mathbf{2}}'\cdot\hat{\mathbf{2}}' = 1$, $\{\hat{\mathbf{1}}', \hat{\mathbf{2}}'\}$ forms an orthonormal basis. Expressing our vector $\mathbf{x}$ under this basis, we have:

$$\begin{aligned}\mathbf{x}(\hat{\mathbf{1}}',\hat{\mathbf{2}}') &= (\mathbf{x}\cdot\hat{\mathbf{1}}')\hat{\mathbf{1}}' + (\mathbf{x}\cdot\hat{\mathbf{2}}')\hat{\mathbf{2}}' \\ &= [\mathbf{x}\cdot(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{1}}' \;+\; [\mathbf{x}\cdot(\sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{2}}' \\ &= [R(\theta)\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})\cdot\hat{\mathbf{1}}]\,\hat{\mathbf{1}}' \;+\; [R(\theta)\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})\cdot\hat{\mathbf{2}}]\,\hat{\mathbf{2}}' \quad \text{by comparision with (4).}\end{aligned}$$

But $\mathbf{x}(\hat{\mathbf{1}}',\hat{\mathbf{2}}')$ is exactly the vector $\mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})$. It is just expressed under the $\{\hat{\mathbf{1}}',\hat{\mathbf{2}}'\}$ basis. Therefore, we deduce that the components of a vector rotated in space about the origin are equal to the components of the vector in the space inversely rotated about the origin. Finally, to verify that we are dealing with the same vector:

$$\begin{aligned}\mathbf{x}(\hat{\mathbf{1}}',\hat{\mathbf{2}}') &= [\mathbf{x}\cdot(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{1}}' \;+\; [\mathbf{x}\cdot(\sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}})]\,\hat{\mathbf{2}}' \\ &= [\mathbf{x}\cdot(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}})]\,(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}}) \;+\; [\mathbf{x}\cdot(\sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}})]\,(\sin\theta\,\hat{\mathbf{1}} + \cos\theta\,\hat{\mathbf{2}}) \\ &= [\mathbf{x}\cdot\hat{\mathbf{1}}]\,\hat{\mathbf{1}} + [\mathbf{x}\cdot\hat{\mathbf{2}}]\,\hat{\mathbf{2}} \\ &= \mathbf{x}(\hat{\mathbf{1}},\hat{\mathbf{2}})\end{aligned}$$

2 Three-dimensional rotations

For the two-dimensional rotations in the $\hat{\mathbf{1}}\hat{\mathbf{2}}$ plane discussed above, it is implied that the axis of rotation lies parallel to the $\hat{\mathbf{3}}$ basis vector, which is oriented perpendicular to the plane. In a rotation in three-dimensions, the axis of rotation can be oriented and located arbitrarily in three dimensions, as shown in Figure 5.

As with (7), we wish to obtain a concise formula for the transformation $G(\theta, \hat{\mathbf{a}}, \mathbf{b})$ which rotates a point's position vector

$$\mathbf{x} \;=\; x\hat{\mathbf{1}} + y\hat{\mathbf{2}} + z\hat{\mathbf{3}}$$

in E^3 to the position vector

$$\mathbf{x}'(\theta, \hat{\mathbf{a}}, \mathbf{b}) \;= G(\theta, \hat{\mathbf{a}}, \mathbf{b})\mathbf{x}$$

The transformation can be found using two approaches. In the first (simpler) approach, the relationships between the different vectors in Figure 5 are analysed graphically. The second approach is a compositing one, in which the transformation is constructed from a sequence of simpler ones. We begin with the graphical approach.

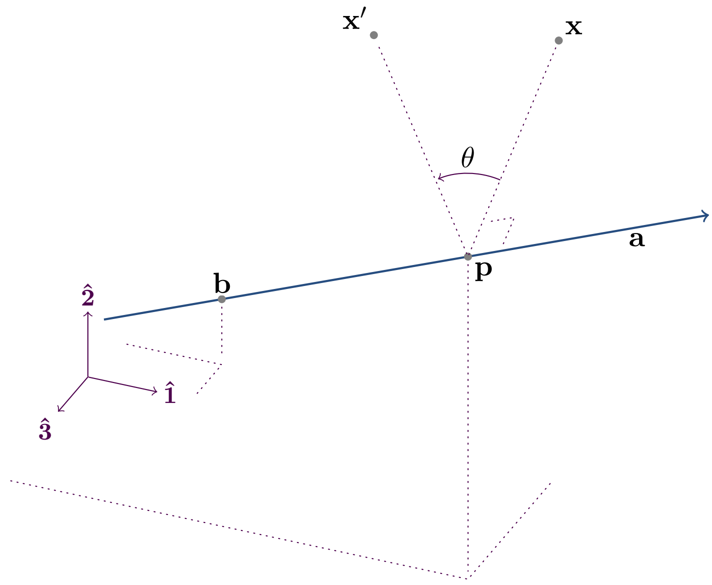


Figure 5: Rotation in three dimensions about an arbitrary axis specified by a point $\mathbf{b}$ lying on the axis and vector $\mathbf{a}$ parallel to the axis.

2.1 Graphical approach

In Figure 5, the vector $\mathbf{a}$, and its normalised form $\hat{\mathbf{a}}$, lie parallel to the axis of rotation. The point $\mathbf{b}$ is specified to lie somewhere on the axis. The pair $(\hat{\mathbf{a}}, \mathbf{b})$ therefore uniquely determines the line of the axis.

The point $\mathbf{p}$ is that point along the axis about which the rotation takes place. By observing that the vector $(\mathbf{p} - \mathbf{b})$ is the component of the vector $(\mathbf{x} - \mathbf{b})$ in the direction $\hat{\mathbf{a}}$, we may write $\mathbf{p} - \mathbf{b} = ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\, \hat{\mathbf{a}}$, because $((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})$ is the magnitude of the projection of $(\mathbf{x} - \mathbf{b})$ in the direction $\hat{\mathbf{a}}$. This gives

$$\mathbf{p} \;=\; \mathbf{b} + ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\, \hat{\mathbf{a}} \tag{10}$$

Next, observe that the points $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{x}'$ are coplanar in the plane formed by sweeping the vector $\mathbf{x} - \mathbf{p}$ to the vector $\mathbf{x}' - \mathbf{p}$. Define two unit vectors

$$\hat{\mathbf{n}} = \frac{1}{|\mathbf{x} - \mathbf{p}|}(\mathbf{x} - \mathbf{p}) \quad \text{and} \quad \hat{\mathbf{m}} = \frac{1}{|\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p})|}(\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p}))$$

By definition, $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = \hat{\mathbf{m}} \cdot \hat{\mathbf{m}} = 1$. And since it is easy to show that for any vectors $\mathbf{u}$ and $\mathbf{v}$ in E^3,

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

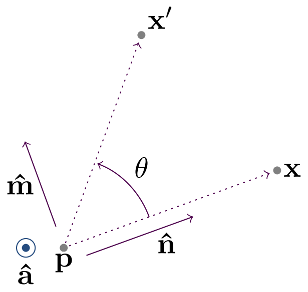
we have $\hat{\mathbf{a}} \cdot \hat{\mathbf{m}} = \hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = 0$. Also,

$$\begin{aligned}
\hat{\mathbf{a}} \cdot \hat{\mathbf{n}} &= \frac{1}{|\mathbf{x} - \mathbf{p}|}\, \hat{\mathbf{a}} \cdot [\mathbf{x} - \mathbf{p}] \\
&= \frac{1}{|\mathbf{x} - \mathbf{p}|}\, \hat{\mathbf{a}} \cdot [\mathbf{x} - \mathbf{b} - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\, \hat{\mathbf{a}}] \\
&= \frac{1}{|\mathbf{x} - \mathbf{p}|}\; [\hat{\mathbf{a}} \cdot (\mathbf{x} - \mathbf{b}) - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} \cdot \hat{\mathbf{a}}] \\
&= 0
\end{aligned}$$

The set $\{\hat{\mathbf{n}}, \hat{\mathbf{m}}, \hat{\mathbf{a}}\}$ therefore forms an orthonormal basis for E^3, and $\{\hat{\mathbf{n}}, \hat{\mathbf{m}}\}$ forms an orthonormal basis for the plane containing $\mathbf{x} - \mathbf{p}$ and $\mathbf{x}' - \mathbf{p}$. Specifically, the vector $\mathbf{x}' - \mathbf{p}$ lying in the plane may be expressed under $\{\hat{\mathbf{n}}, \hat{\mathbf{m}}\}$:

$$\mathbf{x}' - \mathbf{p} \;=\; \xi \hat{\mathbf{n}} + \zeta \hat{\mathbf{m}} \quad \text{for some } \xi, \zeta \in \mathbb{R}$$

What are ξ and ζ? The view of $\mathbf{x} - \mathbf{p}$ and $\mathbf{x}' - \mathbf{p}$ down the length of $\hat{\mathbf{a}}$ is



from which it is evident that

$$\begin{aligned}
(\mathbf{x}' - \mathbf{p}) &= |\mathbf{x}' - \mathbf{p}| \cos\theta \, \hat{\mathbf{n}} \,+\, |\mathbf{x}' - \mathbf{p}| \sin\theta \, \hat{\mathbf{m}} \\
&= |\mathbf{x} - \mathbf{p}| \cos\theta \, \hat{\mathbf{n}} \,+\, |\mathbf{x} - \mathbf{p}| \sin\theta \, \hat{\mathbf{m}} \\
&= |\mathbf{x} - \mathbf{p}| \cos\theta \frac{1}{|\mathbf{x} - \mathbf{p}|} (\mathbf{x} - \mathbf{p}) \,+\, |\mathbf{x} - \mathbf{p}| \sin\theta \, \frac{1}{|\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p})|} (\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p})) \\
&= \cos\theta \, (\mathbf{x} - \mathbf{p}) \,+\, |\mathbf{x} - \mathbf{p}| \sin\theta \, \frac{1}{|\hat{\mathbf{a}} \times (|\mathbf{x} - \mathbf{p}| \, \hat{\mathbf{n}})|} (\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p})) \\
&= \cos\theta \, (\mathbf{x} - \mathbf{p}) \,+\, \sin\theta \, \frac{1}{|\hat{\mathbf{a}} \times \hat{\mathbf{n}}|} (\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p}))
\end{aligned}$$

But since $|\hat{\mathbf{a}} \times \hat{\mathbf{n}}| = |\hat{\mathbf{a}}| \, |\hat{\mathbf{n}}| \sin(\pi/2) = 1$,

$$\mathbf{x}' \,=\, \mathbf{p} \,+\, \cos\theta \, (\mathbf{x} - \mathbf{p}) \,+\, \sin\theta \, (\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p}))$$

Can this be simplified using (10)?

$$\mathbf{x} - \mathbf{p} = \mathbf{x} - \mathbf{b} - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) \, \hat{\mathbf{a}}$$

So

$$\begin{aligned}
\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{p}) &= \hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{b}) - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) \, \hat{\mathbf{a}} \times \hat{\mathbf{a}} \\
&= \hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{b})
\end{aligned}$$

Finally, again using (10),

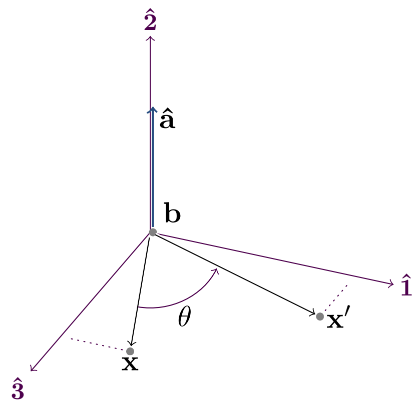
$$\boxed{\begin{aligned}
\mathbf{x}'(\theta, \hat{\mathbf{a}}, \mathbf{b}) &= G(\theta, \hat{\mathbf{a}}, \mathbf{b}) \mathbf{x} \\
&= \mathbf{b} \,+\, ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) \, \hat{\mathbf{a}} \,+\, \cos\theta \, [\mathbf{x} - \mathbf{b} - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) \, \hat{\mathbf{a}}] \,+\, \sin\theta \, [\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{b})]
\end{aligned}} \tag{11}$$

This is a generalisation of a rotation in two dimensions (Eq. (7)) to a rotation in three dimensions.

Does (11) encapsulate the two-dimensional rotations? In the simplest case, suppose that the axis of rotation lies parallel with the $\hat{\mathbf{3}}$ basis vector, so $\hat{\mathbf{a}} = \hat{\mathbf{3}}$, and that the point $\mathbf{b}$ is at the origin, $\mathbf{b} = \mathbf{0}$. A position vector in the $\hat{\mathbf{1}}\hat{\mathbf{2}}$ plane is $\mathbf{x} = x\hat{\mathbf{1}} + y\hat{\mathbf{2}}$. Then using (11),

$$\begin{aligned}
\mathbf{x}'(\theta, \hat{\mathbf{3}}, \mathbf{0}) &= \cos\theta \, [x\hat{\mathbf{1}} + y\hat{\mathbf{2}}] \,+\, \sin\theta \, [-y\hat{\mathbf{1}} + x\hat{\mathbf{2}}] \\
&= [\mathbf{x} \cdot (\cos\theta \, \hat{\mathbf{1}} - \sin\theta \, \hat{\mathbf{2}})] \, \hat{\mathbf{1}} \,+\, [\mathbf{x} \cdot (\sin\theta \, \hat{\mathbf{1}} + \cos\theta \, \hat{\mathbf{2}})] \, \hat{\mathbf{2}}
\end{aligned}$$

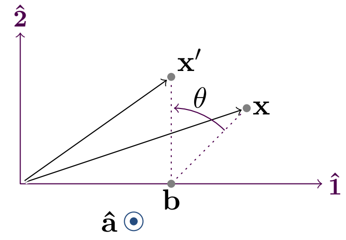
which agrees with (4). Next, suppose that $\hat{\mathbf{a}} = \hat{\mathbf{2}}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{x} = x\hat{\mathbf{1}} + z\hat{\mathbf{3}}$.



Then using (11),

$$\begin{aligned}\mathbf{x}'(\theta, \hat{\mathbf{2}}, \mathbf{0}) &= \cos\theta\,(x\hat{\mathbf{1}} + z\hat{\mathbf{3}}) + \sin\theta\,(z\hat{\mathbf{1}} - x\hat{\mathbf{3}}) \\ &= (\cos\theta\, x + \sin\theta\, z)\,\hat{\mathbf{1}} + (-\sin\theta\, x + \cos\theta\, z)\,\hat{\mathbf{2}}\end{aligned}$$

as expected. Next, suppose that $\hat{\mathbf{a}} = \hat{\mathbf{3}}$, $\mathbf{b} = b\hat{\mathbf{1}}$ and $\mathbf{x} = x\hat{\mathbf{1}} + y\hat{\mathbf{2}}$.

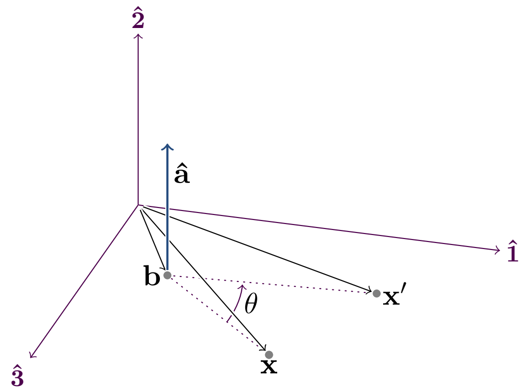


Then using (11),

$$\begin{aligned}\mathbf{x}'(\theta, \hat{\mathbf{3}}, b\hat{\mathbf{1}}) &= b\hat{\mathbf{1}} \;+\; \cos\theta\,[(x-b)\hat{\mathbf{1}} + y\hat{\mathbf{2}}] \;+\; \sin\theta\,[-y\hat{\mathbf{1}} + (x-b)\hat{\mathbf{2}}] \\ &= [b \;+\; \cos\theta\,(x-b) - \sin\theta\, y]\,\hat{\mathbf{1}} + [\cos\theta\, y + \sin\theta\,(x-b)]\,\hat{\mathbf{2}}\end{aligned}$$

Specifically, for $\mathbf{x} = 3\hat{\mathbf{1}} + \hat{\mathbf{2}}$, $\mathbf{x}'(\frac{\pi}{4}, \hat{\mathbf{3}}, 2\hat{\mathbf{1}}) \;=\; 2\hat{\mathbf{1}} \;+\; \sqrt{2}\,\hat{\mathbf{2}}$, as expected.

Next, suppose that $\hat{\mathbf{a}} = \hat{\mathbf{2}}$, $\mathbf{b} = 1\hat{\mathbf{1}} + 2\hat{\mathbf{3}}$, $\theta = \frac{\pi}{4}$ and $\mathbf{x} = 3\hat{\mathbf{1}} + 4\hat{\mathbf{3}}$.



Using (11),

$$\begin{aligned}\mathbf{x}'(\pi/4, \hat{\mathbf{2}}, 1\hat{\mathbf{1}} + 2\hat{\mathbf{3}}) &= \hat{\mathbf{1}} + 2\hat{\mathbf{3}} + \cos(\pi/4)[2\hat{\mathbf{1}} + 2\hat{\mathbf{3}}] + \sin(\pi/4)[2\hat{\mathbf{1}} - 2\hat{\mathbf{3}}] \\ &= 2\hat{\mathbf{1}} + \sqrt{2}\,\hat{\mathbf{3}}\end{aligned}$$

2.2 Compositing approach

In a compositing approach to deriving (11), the desired three-diminensional transformation is constructed from a sequence of seven simpler ones. We begin by reproducing the schematic layout shown in Figure 5. Vector $\mathbf{a}$ and point $\mathbf{b}$ define an axis of rotation. As before, we wish to rotate any point $\mathbf{x}$ by angle θ about axis to obtain $\mathbf{x}'$.

Step 1—Translate by $-\mathbf{p}$ so that $\mathbf{p}$ moves to origin. Given the expression for the point $\mathbf{p}$ in (10), the translation transformation is

$$T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x}) \;:\; \mathbf{y} \mapsto T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\mathbf{y} = \mathbf{y} - \mathbf{b} - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\,\hat{\mathbf{a}} \qquad \text{for any } \mathbf{y} \in E^3 \tag{12}$$

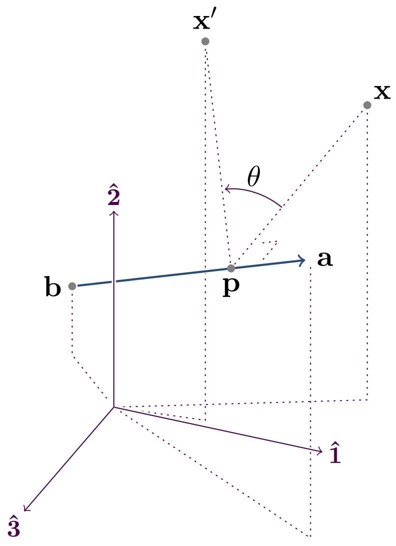
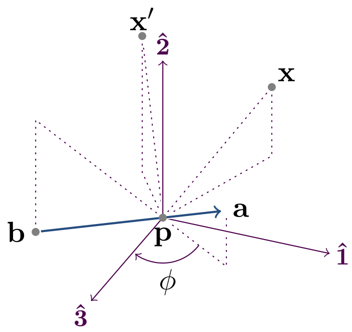


Figure 6: Points $\mathbf{x}$ and $\mathbf{x}'$, vector $\mathbf{a}$, point $\mathbf{b}$ and angle θ.



Step 2—Rotate about $\hat{\mathbf{2}}$ s.t. a_1 vanishes, i.e., s.t. $\mathbf{a} \perp \hat{\mathbf{1}}$. To rotate about $\hat{\mathbf{2}}$ by angle $(-\phi)$, (5) suggests the use of the rotation transformation

$$R_1(\phi) \,:\, \mathbf{y} \mapsto R_1(\phi)\mathbf{y} \tag{13}$$

where $R_1(\phi)$ may be represented by the matrix

$$R_1(\phi) \;=\; \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$

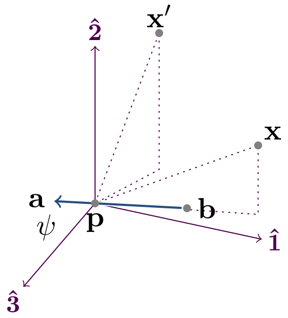
and since ϕ is the angle between $\hat{\mathbf{3}}$ and the projection of $\hat{\mathbf{a}}$ onto the $\hat{\mathbf{1}}\hat{\mathbf{3}}$ plane,

$$\cos\phi = \frac{a_3}{\sqrt{a_1^2 + a_3^2}} \;, \quad \sin\phi = \frac{a_1}{\sqrt{a_1^2 + a_3^2}} \tag{14}$$

Applying the transformation to $\hat{\mathbf{a}}$ gives:

$$\begin{aligned} R_1(\phi)\hat{\mathbf{a}} \;&= (\cos\phi a_1 - \sin\phi a_3)\hat{\mathbf{1}} + a_2\hat{\mathbf{2}} + (\sin\phi a_1 + \cos\phi a_3)\hat{\mathbf{3}} \\ &= \frac{a_3 a_1 - a_1 a_3}{\sqrt{a_1^2 + a_3^2}}\,\hat{\mathbf{1}} + a_2\hat{\mathbf{2}} + \frac{a_1^2 + a_3^2}{\sqrt{a_1^2 + a_3^2}}\,\hat{\mathbf{3}} \\ &= a_2\hat{\mathbf{2}} + \sqrt{a_1^2 + a_3^2}\,\hat{\mathbf{3}} \end{aligned} \tag{15}$$

so that, as expected, $(R_1(\phi)\hat{\mathbf{a}}) \cdot \hat{\mathbf{1}}$ vanishes. The transformed vector $R_1(\phi)\hat{\mathbf{a}}$ lies in the $\hat{\mathbf{2}}\hat{\mathbf{3}}$ plane.



Step 3—Rotate about $\hat{\mathbf{1}}$ s.t. a_2 vanishes, i.e., s.t. $\hat{\mathbf{a}} = \hat{\mathbf{3}}$. To rotate about $\hat{\mathbf{1}}$ by angle $(-\psi)$ such that a_2 vanishes, the rotation transformation is

$$R_2(\psi) \,:\, \mathbf{y} \mapsto R_2(\psi)\mathbf{y} \tag{16}$$

where $R_2(\psi)$ may be represented by the matrix

$$R_2(\psi) \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

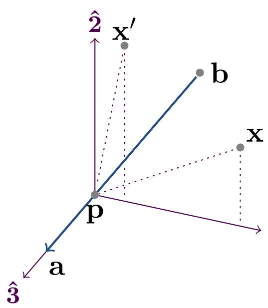
The angle ψ is the angle between $\hat{\mathbf{3}}$ and the projection of $\mathbf{a}$ onto the $\hat{\mathbf{2}}\hat{\mathbf{3}}$ plane. That projection vector is given by (15). So

$$\cos\psi = \sqrt{a_1^2 + a_3^2}\,, \quad \sin\psi = a_2 \tag{17}$$

Applying the transformation to $R_1(\phi)\hat{\mathbf{a}}$ gives:

$$\begin{aligned} R_2(\psi)R_1(\phi)\hat{\mathbf{a}} &= R_2(\psi)(a_2\hat{\mathbf{2}} + \sqrt{a_1^2 + a_3^2}\,\hat{\mathbf{3}}) \\ &= \left(a_2\cos\psi - \sqrt{a_1^2 + a_3^2}\sin\psi\right)\hat{\mathbf{2}} + \left(a_2\sin\psi + \sqrt{a_1^2 + a_3^2}\cos\psi\right)\hat{\mathbf{3}} \\ &= \left(a_2\sqrt{a_1^2 + a_3^2} - \sqrt{a_1^2 + a_3^2}\,a_2\right)\hat{\mathbf{2}} + \left(a_2^2 + (a_1^2 + a_3^2)\right)\hat{\mathbf{3}} \\ &= \hat{\mathbf{3}} \end{aligned}$$

so that, as expected, $(R_2(\psi)R_1(\phi)\hat{\mathbf{a}}) \cdot \hat{\mathbf{2}}$ vanishes.

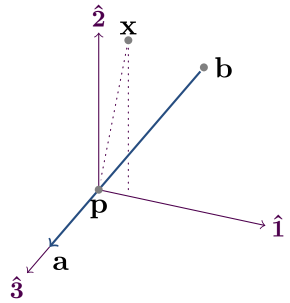


Step 4—Rotate about $\hat{\mathbf{a}} = \hat{\mathbf{3}}$ by angle θ. The rotation transformation is

$$R_3(\theta) \,:\, \mathbf{y} \mapsto R_3(\theta)\mathbf{y} \tag{18}$$

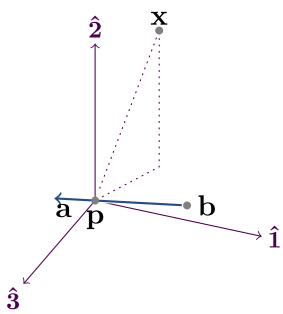
where $R_3(\theta)$ may be represented by the matrix

$$R_3(\theta) \;=\; \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



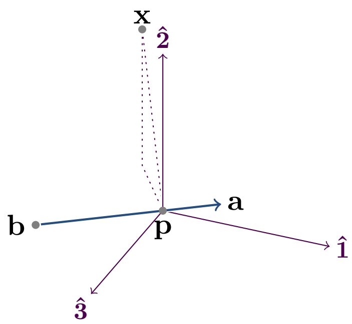
Step 5—Rotate back about $\hat{\mathbf{1}}$. The rotation transformation is

$$R_2(-\psi)\ :\ \mathbf{y} \mapsto R_2(-\psi)\mathbf{y} \tag{19}$$



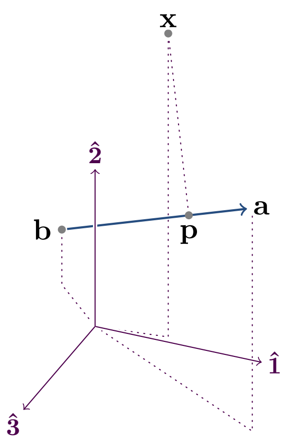
Step 6—Rotate back about $\hat{\mathbf{2}}$. The rotation transformation is

$$R_1(-\phi)\ :\ \mathbf{y} \mapsto R_1(-\phi)\mathbf{y} \tag{20}$$



Step 7—Translate back. The translation transformation is

$$T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x})\ :\ \mathbf{y} \mapsto T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x})\mathbf{y} = \mathbf{y} + \mathbf{b} + ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}})\,\hat{\mathbf{a}} \tag{21}$$



The composition of transformations is

$$C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\ :\ \mathbf{y} \mapsto C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\,\mathbf{y} \quad \text{for any } \mathbf{y} \in E^3 \tag{22}$$

with

$$C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x}) = T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x}) R_1(-\phi(\hat{\mathbf{a}})) R_2(-\psi(\hat{\mathbf{a}})) R_3(\theta) R_2(\psi(\hat{\mathbf{a}})) R_1(\phi(\hat{\mathbf{a}})) T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x}) \tag{23}$$

and with $\phi(\hat{\mathbf{a}})$ and $\psi(\hat{\mathbf{a}})$ given by (14) and (17).

For the moment, consider just the rotational transformations. Write

$$C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x}) = T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x}) A(\theta, \phi(\hat{\mathbf{a}}), \psi(\hat{\mathbf{a}})) T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x}) \tag{24}$$

with

$$A(\theta, \phi, \psi) = R_1(-\phi) R_2(-\psi) R_3(\theta) R_2(\psi) R_1(\phi)$$

It is easy to show that

$$\begin{aligned}
R_2(\psi) R_1(\phi) &= \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ -\sin\phi\sin\psi & \cos\psi & -\cos\phi\sin\psi \\ \sin\phi\cos\psi & \sin\psi & \cos\phi\cos\psi \end{bmatrix} \\
R_1(-\phi) R_2(-\psi) &= \begin{bmatrix} \cos\phi & -\sin\phi\sin\psi & \sin\phi\cos\psi \\ 0 & \cos\psi & \sin\psi \\ -\sin\phi & -\cos\phi\sin\psi & \cos\phi\cos\psi \end{bmatrix} \\
R_3(\theta) R_2(\psi) R_1(\phi) &= \begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi\sin\psi & -\sin\theta\cos\psi & -\cos\theta\sin\phi + \sin\theta\cos\phi\sin\psi \\ \sin\theta\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\theta\cos\psi & -\sin\theta\sin\phi - \cos\theta\cos\phi\sin\psi \\ \sin\phi\cos\psi & \sin\psi & \cos\phi\cos\psi \end{bmatrix}
\end{aligned}$$

After some algebraic manipulation, and applying the definitions of (14) and (17), we arrive at

$$\begin{aligned}
A(\theta, \phi, \psi) &= A(\theta, \hat{\mathbf{a}}) \\
&= \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix} + \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \sin\theta - \begin{bmatrix} a_1^2 - 1 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 - 1 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 - 1 \end{bmatrix} \cos\theta
\end{aligned}$$

The ij-the matrix element may be written as

$$[A(\theta, \hat{\mathbf{a}})]_{ij} = a_i a_j + \epsilon_{ikj} a_k \sin\theta + (a_i a_j - \delta_{ij}) \cos\theta \tag{25}$$

where

$$\delta_{ij} \equiv \begin{cases} 1, & \text{for } i = j \\ \text{otherwise} \end{cases}$$

$$\epsilon_{ijk} \equiv \begin{cases} 0, & \text{for } i = j \text{ or } i = k \text{ or } j = k. \\ 1, & \text{for any even permutation of } ijk, \text{ e.g., } 123. \\ -1, & \text{otherwise, e.g., } 132. \end{cases}$$

and where summation over the index k is assumed.

Next, the translation transformations. The i-th components of $T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\mathbf{y}$ (Eq. (12)) and $T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x})$ (Eq. (21)) are

$$\begin{aligned}
[T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\mathbf{y}]_i &= (\delta_{ij} - a_i a_j)(x_j - b_j) + y_i - x_i \\
[T(-\hat{\mathbf{a}}, -\mathbf{b}, -\mathbf{x})\mathbf{y}]_i &= -(\delta_{ij} - a_i a_j)(x_j - b_j) + y_i + x_i
\end{aligned} \tag{26}$$

Using (25) and (26), the i-th component of $C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\,\mathbf{x}$ in (22) is

$$\begin{aligned}
[C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\,\mathbf{x}]_i &= -\,(\delta_{ij} - a_i a_j)(x_j - b_j) + x_i + [A(\theta, \hat{\mathbf{a}}) T(\hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\mathbf{x}]_i \\
&= -\,(\delta_{ij} - a_i a_j)(x_j - b_j) + x_i \\
&\quad + [a_i a_j + \epsilon_{ikj} a_k \sin\theta + (a_i a_j - \delta_{ij}) \cos\theta][(\delta_{jl} - a_j a_l)(x_l - b_l)]
\end{aligned}$$

Finally, by expanding the terms,

$$\begin{aligned}
[C(\theta, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{x})\,\mathbf{x}]_i &= b_i + (x_k - b_k) a_k a_i + \cos\theta (x_i - b_i - (x_k - b_k) a_k a_i) + \sin\theta \epsilon_{ikl} a_k (x_l - b_l) \\
&= b_i \,+\, ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) a_i \,+\, \cos\theta\,[(\mathbf{x} - \mathbf{b})_i - ((\mathbf{x} - \mathbf{b}) \cdot \hat{\mathbf{a}}) a_i] \,+\, \sin\theta\,[\hat{\mathbf{a}} \times (\mathbf{x} - \mathbf{b})]_i
\end{aligned}$$

which is exactly the i-th component in (11).